Nilpotent orbits of a generalization of Hodge structures

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Abstract

We study a generalization of Hodge structures which first appeared in the work of Cecotti and Vafa. It consists of twistors, that is, holomorphic vector bundles on $\mathbb{P}^1$, with additional structure, a flat connection on $\mathbb{C}^*$, a real subbundle and a pairing. We call these objects TERP-structures. We generalize to TERP-structures a correspondence of Cattani, Kaplan and Schmid between nilpotent orbits of Hodge structures and polarized mixed Hodge structures. The proofs use work of Simpson and Mochizuki on variations of twistor structures and a control of the Stokes structures of the poles at zero and infinity. The results are applied to TERP-structures which arise via oscillating integrals from holomorphic functions with isolated singularities.

Contents

1 Introduction

This paper studies generalizations of Hodge structures and variations of them. They appeared in the work of Cecotti and Vafa ([CV91][CV93]) on supersymmetric field theories. The abstract notion is of general nature. It is studied under the name TERP-structure in [Her03].

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In this paper we continue to investigate such TERP-structures. More specifically, we study the relation between nilpotent orbits of them and a corresponding generalization of mixed Hodge structures. This result is an extension of a similar correspondence in Hodge theory due to Cattani, Kaplan and Schmid.

Closely related objects, called twistor structures (which also generalize Hodge structures) appeared in the work of Simpson ([Sim97]). TERP-structures are enriched twistors. A twistor is simply a holomorphic vector bundle $\widehat{H}$ on $\mathbb{P}^1$. A twistor $\widehat{H}$ is called pure of weight $w \in \mathbb{Z}$ if it is semi-stable of slope w. A pure twistor of weight 0 is polarized if there is a sesquilinear pairing $\widehat{S} : \widehat{H}_z \times \widehat{H}_{-1/\overline{z}} \to \mathbb{C}$ for all $z \in \mathbb{P}^1$ such that the induced pairing on $\Gamma(\mathbb{P}^1, \widehat{\mathcal{H}})$ is hermitian and positive definite. This can be generalized to any weight. Note that a single twistor is a rather elementary object. However, if one takes into account parameters, then the resulting structure is quite involved, a variation of pure polarized twistors is actually equivalent to a harmonic bundle on the parameter space (see [Sim88][Sim90][Sim92]).

A TERP-structure (the name stems from "T" for twistor, "E" for extension, "R" for real structure and "P" for pairing) is a twistor with additional data which generalize all ingredients of a polarized Hodge structure, that is, a real structure, a pairing and the Hodge filtration. More precisely, a TERP-structure of weight 0 is a holomorphic vector bundle H on $\mathbb{C}$ with a flat connection ∇ on $H_{|\mathbb{C}^*}$ which has a pole of order at most two at 0. The real structure consists of a flat real subbundle $H'_{\mathbb{R}}$ on $\mathbb{C}^*$. The pairing in this case is a pairing $P : H_z \times H_{-z} \to \mathbb{C}$ for $z \in \mathbb{C}$, which is symmetric and nondegenerate on $\mathbb{C}$ and flat on $\mathbb{C}^*$ and which takes values in $\mathbb{R}$ on the real subbundle. In many applications it makes sense to consider the data $(H_{|\mathbb{C}^*}, \nabla, H'_{|\mathbb{R}}, P)$ as "topological objects" and the extension of the bundle to 0 as transcendent. This extension is the generalization of the Hodge filtration. A key point now is that real and flat structures allow to construct canonically an extension to infinity making up a twistor (see chapter 3 for the precise construction). The extension at ∞ then generalizes naturally the complex conjugate of the Hodge filtration. By construction, the resulting twistor comes equipped with a meromorphic connection with poles of order at most two at zero and infinity. Moreover, the given pairing P and the construction of the extension yield a pairing $\widehat{S}$ like the one described above.

A TERP-structure H will be called pure (resp. polarized pure) if the corresponding object $(\widehat{H}, \widehat{S})$ is a pure (resp. polarized pure) twistor. Pureness generalizes the notion of opposite filtrations, i.e. of a Hodge structure. Polarized pure TERP-structures generalize polarized pure Hodge structures. The precise relation between TERP-structures and twistor structures is reviewed in chapter 3. We also make some comments on the parameter case, i.e., variations of TERP- resp. twistor structures, which were treated in great detail in [Her03].

Important tools in the study of variations of Hodge structures are the notions of nilpotent orbits of Hodge structures and mixed Hodge structures, the first one being geometric, the second linear. There is a beautiful correspondence relating them (theorem 2.5). One direction is due to Schmid [Sch73], the other one has been treated later in a series of papers ([CK82][CKS86][CK89]) by Cattani, Kaplan and Schmid. We give a short reminder of these notions and results in chapter 2.

The main purpose of the present paper is the generalization of this correspondence to TERP-structures. The notion of a nilpotent orbit has a rather simple generalization (definition 4.1): We say that a TERP-structure $(H, \nabla, H'_{\mathbb{R}}, P)$ induces a nilpotent orbit if $\pi_r^*(H, \nabla, H'_{\mathbb{R}}, P)$ is a polarized pure TERP-structure for any $r \in \mathbb{C}^*$ with $|r| \ll 1$; here $\pi_r : \mathbb{C} \to \mathbb{C}, z \mapsto z \cdot r$, for $r \in \mathbb{C}^*$. It comes as a surprise that a simple rescaling of the coordinate on $\mathbb{C}$ changes the twistor which results from the gluing procedure in an essential way.

The generalization of a polarized mixed Hodge structure is what we call a mixed TERP-structure. Its definition (definition 9.3) is rather involved, as we have to deal with a possibly irregular singularity of the connection ∇ at zero. Let us first explain the regular singular case. A TERP-structure $(H, \nabla, H'_{\mathbb{R}}, P)$ is called regular singular if the pole at 0 is so. In that case there is a well known procedure to obtain a filtration on the space H^∞ of multivalued flat global sections of the bundle $H_{|\mathbb{C}^*}$. Up to a twist, it was first considered by Varchenko [Var80] in the context of hypersurface singularities, and refined later in [SS85][Sai89]. For a regular singular TERP-structure, the condition to be mixed simply means that this filtration is part of a polarized mixed Hodge structure. If, however, the pole at zero becomes irregular, then we have to modify this condition. First, we require that the formal decomposition of (H, ∇) can be done without ramification. Moreover, we need a compatibility condition between $H'_{\mathbb{R}}$ and the Stokes structure defined by the irregular pole. Under these two hypotheses, the regular singular factors which appear in the formal decomposition are themselves TERP-structures and the main condition imposed is that they induces polarized mixed Hodge structures as before. The details are explained in the chapters 8 and 9. With this notion in mind, we can state the generalized correspondence as follows.

Conjecture 1.1. *(conjecture 9.2) A TERP-structure which does not require a ramification is a mixed TERP-structure iff it induces a nilpotent orbit.*

The main result of this paper is a proof of a good part of this conjecture, namely:

Theorem 1.2. *(theorem 9.3)*

1. *The conjecture is true if the TERP-structure is regular singular.*
2. *The implication $\Rightarrow$ is true for any TERP-structure.*

The implication $\Rightarrow$ in the regular singular case was shown in [Her03, theorem 7.20], using the analogous implication in the correspondence between polarized mixed Hodge structures and nilpotent orbits of Hodge structures. The opposite implication $\Leftarrow$ for regular singular TERP-structures is proved in chapter 6 and uses quite different techniques, namely, it relies on a recent result of Mochizuki [Moc07]: In that paper, he constructs, among other things, for any tame harmonic bundle on the complement of a normal crossing divisor a limit polarized mixed twistor structure. This generalizes Schmid's limit polarized mixed Hodge structure defined by a variation of Hodge structures.
For nilpotent orbits of TERP-structures, we only need the one-variable version of the limit mixed twistor structure ([Moc07, theorem 12.1]). To apply it, we establish in chapter 5 a correspondence (lemma 5.9) between certain integrable polarized mixed twistor structures and polarized mixed Hodge structures equipped with a semi-simple automorphism. This correspondence extends similar correspondences in [Moc07, ch. 3].
The last part of the paper (chapters 8 to 10) deals with the general (i.e., irregular) case. The implication $\Rightarrow$ in general is proved in chapter 9. It combines the regular singular case with a discussion of the Stokes structure. In the end it comes down to a Riemann boundary value problem which we are able to solve by an argument involving the Birkhoff decomposition of the loop group $\Lambda GL_n(\mathbb{C})$ [PS86, (8.1.2)].
A particular case arises if the pole part of the connection is a semi-simple endomorphism with pairwise different eigenvalues. Such TERP-structures are called semi-simple and the implication $\Rightarrow$ was already established by Dubrovin [Dub93, proposition 2.2] in that case.

For semi-simple TERP-structures of rank two, families $\bigcup_{r>0}\pi_{r^{-1}}^*(H,\nabla,H'_\mathbb{R},P)$ of TERP-structures are closely related to solutions of the sinh-Gordon equation $(\partial_r^2+\frac{1}{r}\partial_r)u(r)=\sinh u(r)$ [CV91][CV93][Dub93] (and implicitly also [IN86]). The implication $\Leftarrow$ in the semi-simple rank two case is equivalent to the claim that the only solutions $u(r)$ which are smooth and real for large r are the one parameter family of solutions which were studied in [MTW77]. This is very probably true, but the statements in [IN86][MTW77] do not imply it immediately. See the discussion at the end of chapter 10 for more details. The conjecture in general predicts that the singularity freeness for large r of certain systems of differential equations is equivalent to "linear" conditions which are neatly formulated in terms of the Stokes data.
The semi-simple case is also interesting because of the apparent simplicity in which the structure can be encoded.

Lemma 1.3. *(lemma 10.1) Any data $(w,u_1,...,u_n,\xi,T)$ where $w\in\mathbb{Z}$, $u_i\in\mathbb{C}$ with $u_i\neq u_j$ for $i\neq j$, $\xi\in S^1$ with $\Re(\frac{u_i-u_j}{\xi})<0$ for $i<j$ and $T\in M(n\times n,\mathbb{R})$ upper triangular with $T_{ii}=1$ give rise to a unique semi-simple mixed TERP-structure of weight w. The numbers $u_1,...,u_n$ are the eigenvalues of the pole part of the connection and the Stokes structure is equivalent to the matrix T together with the point ξ. Any semi-simple mixed TERP-structure arises in such a way (the choice of ξ is not unique and influences T).*

An interesting question is to know which of these TERP-structures are pure and polarized. The following conjecture proposes a partial answer.

Conjecture 1.4. *(see conjecture 10.2) If the matrix $T+T^{tr}$ is positive definite, then the TERP-structure is pure and polarized.*

The conjecture is proved in chapter 10 for Stokes data which are related to the ADE-singularities. One ingredient are beautiful results in [Del] and [Loo74] which roughly say that the semi-universal unfolding of an ADE-singularity is a one to one atlas for all distinguished bases up to signs and a finite to one atlas for possible Stokes data. Another ingredient is a recent fundamental result of Sabbah [Sab05a, theorem 4.9] on TERP-structures for tame functions, which applies to the ADE-singularities and their unfoldings.
Using oscillating integrals resp. by the Fourier-Laplace transform of its Gauss-Manin system, any function germ $f:(\mathbb{C}^w,0)\to(\mathbb{C},0)$ with an isolated singularity at zero and similarly any tame function $f:Y\to\mathbb{C}$ on an

affine manifold Y gives rise to a mixed TERP-structure (theorem 11.1). This result relies on the work of many different people. In the tame case, it has been established (although it is not expressed in these terms) in [Sab] and [DS03]. For function germs, the construction is described in [Her03, 8.1].
Sabbah's new result [Sab05a, theorem 4.9] (see theorem 11.2) states that for tame functions, this TERP-structure is always pure and polarized. It seems to be of fundamental importance for future study of tame functions. In a sense, it should be seen as the analogue to the fact that the primitive part of the cohomology of a compact Kähler manifold is a sum of polarized pure Hodge structures. Even though this analogy is only a sort of meta-theorem, it can hopefully be turned into more concrete results using the point of view of mirror symmetry: certain tame functions on affine manifolds correspond to certain Fano manifolds. We give some speculations in that direction in chapter 11.
TERP-structures for tame functions, and even Sabbah's theorem, are treated from the physicists' perspective in [CV91][CV93]. The functions are part of Landau-Ginzburg models, the positive definite hermitian metric of the polarized pure TERP-structure is a ground state metric. Also lemma 1.3 is already used implicitly in [CV91][CV93]. Semi-simple TERP-structures arises from massive field theories, and it is an important feature in cit.loc. that these can be encoded by simple data as in lemma 1.3.
It is an elementary computation that the TERP-structure $TERP(f)$ of a function f (a germ or tame) satisfies $TERP(r \cdot f) = \pi^*_{r^{-1}}(TERP(f))$ for any parameter $r \in \mathbb{C}^*$. The fact that $TERP(f)$ is mixed and theorem 9.3 2. imply that $TERP(r \cdot f)$ is pure and polarized for $|r| \gg 0$. This proves the main part of the conjecture 8.3 in [Her03]. It endows a part of the semi-universal unfolding space with a positive definite hermitian metric. This will hopefully have applications for moduli space questions or Torelli problems.
In [CV91][CV93] the limit $r \to \infty$ is called infrared limit, the limit $r \to 0$ is called ultraviolet limit. We also consider $r \to 0$ and define a counterpart of a nilpotent orbit which is called Sabbah orbit (definition 4.1). The reason for this is that Sabbah has defined in [Sab] for any (not necessarily regular singular) TERP-structure a filtration $F^\bullet_{Sab}$ on H^∞. In particular, it was proved that this filtration gives a mixed Hodge structure for TERP-structures coming from tame functions. However, polarizations were not considered in that paper. The "Sabbah orbit"-version of our correspondence reads as follows.

Theorem 1.5. *(see theorem 7.3) A TERP-structure induces a Sabbah orbit iff a twisted version of the filtration $F^\bullet_{Sab}$ gives rise to a polarized mixed Hodge structure.*

The implication $\Leftarrow$ is analogous to [Her03, theorem 7.20], the implication $\Rightarrow$ uses again [Moc07, theorem 12.1]. This theorem 1.5 and [Sab05a, theorem 4.9] show that $F^\bullet_{Sab}$, twisted appropriately, even makes up a polarized mixed Hodge structure.
Besides [CV91][CV93], this paper owes a lot to [Sab05b] and [Moc07]. In [Sab05b], Simpson's notion of a variation of twistor structures is generalized to polarizable twistor $\mathcal{D}$-modules. Chapter 7 of cit.loc. is devoted to twistor structures with connection on $\mathbb{C}^*$ and a flat hermitian pairing $\widehat{S}$ as above (one might call them "TEH"-structures). The extraordinary long paper [Moc07] provided us with a central degeneration result used in the proofs of theorem 6.6 and theorem 7.3. In the third chapter of cit.loc., nilpotent orbits of polarized twistor structures are considered, however, these twistors are not equipped with connections on $\mathbb{C}^*$, so they are less rich than TERP-structures.

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Notations: For any complex manifold M, we denote by $\overline{M}$ the same real manifold, equipped with the conjugate complex structure, i.e., $\mathcal{O}_{\overline{M}} := \overline{\mathcal{O}}_M$. We will need at different places a total ordering on $\mathbb{C}$ extending the usual real ordering of $\mathbb{R}$. This will be the lexicographic one, that is, we will write $\alpha < \beta$ iff either $\Re(\alpha) < \Re(\beta)$ or $\Re(\alpha) = \Re(\beta)$ and $\Im(\alpha) < \Im(\beta)$. However, the interval notation $[\alpha, \alpha']$ for $\alpha, \alpha' \in \mathbb{R}$ will continue to denote all $\beta \in \mathbb{R}$ with $\alpha \leq \beta \leq \alpha'$. If H is a holomorphic vector bundle on a complex manifold M, we write $H \in VB_M$ for short and we will use the symbol $\mathcal{H}$ to denote its sheaf of holomorphic sections. In [Her03] pure TERP-structures and their variations were called (tr.TERP)-structures, polarized pure TERP-structures and their variations were called (pos.def.tr.TERP)-structures. The following maps will be used frequently in the paper.

$$i : \mathbb{C}^* \hookrightarrow \mathbb{C} \ ; \quad \widetilde{i} : \mathbb{C}^* \hookrightarrow \mathbb{P}^1 \backslash \{0\} \ ; \quad \widehat{i} : \mathbb{C}^* \hookrightarrow \mathbb{P}^1 \ ; \quad \pi_r : \mathbb{C} \to \mathbb{C} \ , \ z \mapsto r \cdot z \text{ for } r \in \mathbb{C}^*;$$

$$j : \mathbb{P}^1 \to \mathbb{P}^1 \ , \ j(z) = -z \ ; \quad \gamma : \mathbb{P}^1 \to \mathbb{P}^1 \ , \ \gamma(z) = \overline{z^{-1}} \ ; \quad \sigma : \mathbb{P}^1 \to \mathbb{P}^1 \ , \ \sigma(z) = -\overline{z^{-1}}.$$

2 Polarized mixed Hodge structures and nilpotent orbits of Hodge structures

This chapter recalls for the reader's convenience some classical notions from Hodge theory. We give the definition of nilpotent orbits and state the correspondence between nilpotent orbits of polarized Hodge structures and (limit) polarized mixed Hodge structures. This correspondence is one of the main motivation of our work. Throughout the whole chapter, w will be an integer, H a complex vector space of finite dimension, $H_{\mathbb{R}}$ a real subspace with $H = H_{\mathbb{R}} \oplus iH_{\mathbb{R}}$, and S a nondegenerate $(-1)^w$-symmetric pairing on H with real values on $H_{\mathbb{R}}$.

Definition 2.1. *A polarized Hodge structure of weight $w \in \mathbb{Z}$ (abbreviation PHS) consists of data $H, H_{\mathbb{R}}$ and S as above and an exhaustive decreasing Hodge filtration $F^\bullet$ on H with the following properties.*

$$\begin{aligned} F^p \oplus \overline{F^{w+1-p}} &= H, && (2.1)\\ S(F^p, F^{w+1-p}) &= 0, && (2.2)\\ i^{p-(w-p)} \cdot S(a, \overline{a}) &> 0 \quad \text{for } a \in \left(F^p \cap F^{w+1-p}\right) \setminus \{0\}. && (2.3) \end{aligned}$$

Remarks:

1. A tuple $(H, H_{\mathbb{R}}, F^\bullet, w)$ with (2.1) is a pure Hodge structure of weight w.
2. The filtrations $F^\bullet$ and $\overline{F^{w-\bullet}}$ are called opposite if they satisfy (2.1). This condition is equivalent to the Hodge decomposition $H = \bigoplus_p H^{p,w-p}$, where $H^{p,w-p} := F^p \cap \overline{F^{w-p}}$. Of course, $\overline{H^{p,q}} = H^{q,p}$.
3. Given a PHS, the Hodge subspaces $H^{p,w-p}$ are orthogonal with respect to the pairing $S(\ , \overline{\cdot})$ The pairing

$$\begin{aligned} h : H \times H &\to \mathbb{C},\\ (a, b) &\mapsto i^{p-(w-p)} \cdot S(a, \overline{b}) \quad \text{for } a \in H^{p,w-p}, b \in H \end{aligned}$$

   is hermitian and positive definite. This pairing distinguishes a PHS from a Hodge structure.

Mixed Hodge structures have been introduced by Deligne [Del71] in order to study the cohomology of singular or non-compact Khler manifolds. A MHS contains a second filtration called weight filtration such that the Hodge filtration induces pure Hodge structures of appropriate weights on the graded spaces with respect to the weight filtration. In Schmid's work, a more specific variant of mixed Hodge structures is considered: The weight filtration is always induced by a given nilpotent endomorphism on $H_{\mathbb{R}}$ satisfying

$$S(N\, a, b) + S(a, N\, b) = 0, \qquad (2.4)$$

i.e., which is an infinitesimal isometry of S. These data yield a weight filtration $W_\bullet$ in the following way.

Lemma 2.2. *[Sch73, Lemma 6.4] Let $(H, H_{\mathbb{R}}, S, N, w)$ be as above.*

1. *There exists a unique exhaustive increasing filtration $W_\bullet$ on $H_{\mathbb{R}}$ such that $N(W_l) \subset W_{l-2}$ and such that $N^l : Gr^W_{w+l} \to Gr^W_{w-l}$ is an isomorphism.*
2. *The filtration satisfies $S(W_l, W_{l'}) = 0$ for $l + l' < w$.*
3. *A nondegenerate $(-1)^{w+l}$-symmetric bilinear form S_l is well defined on Gr^W_{w+l} for $l \geq 0$ by $S_l(a, b) := S(\widetilde{a}, N^l \widetilde{b})$ for $a, b \in Gr^W_{w+l}$ with representatives $\widetilde{a}, \widetilde{b} \in W_{w+l}$.*
4. *The primitive subspace $P_{w+l} \subset Gr^W_{w+l}$ is defined by*

$$P_{w+l} := \ker(N^{l+1} : Gr^W_{w+l} \to Gr^W_{w-l-2})$$

   for $l \geq 0$ and by $P_{w+l} := 0$ for $l < 0$. Then

$$Gr^W_{w+l} = \bigoplus_{i \geq 0} N^i P_{w+l+2i}, \qquad (2.5)$$

   and this decomposition is orthogonal with respect to S_l if $l \geq 0$.

Definition 2.3. *[CK82][Her02] A polarized mixed Hodge structure of weight w (abbreviation PMHS) consists of data $H, H_{\mathbb{R}}, S, N$ and $W_\bullet$ as above and an exhaustive decreasing Hodge filtration $F^\bullet$ on H with the following properties.*

1. *The filtration $F^\bullet Gr_k^W$ on Gr_k^W gives a pure Hodge structure of weight k,*
2. *N is a $(-1,-1)$-morphism of mixed Hodge structures,*
$$N(F^p) \subset F^{p-1}, \tag{2.6}$$
3. $$S(F^p, F^{w+1-p}) = 0. \tag{2.7}$$
4. *For $a \in \left(F^p P_{w+l} \cap \overline{F^{w+l-p} P_{w+l}}\right) \setminus \{0\}$*
$$i^{p-(w+l-p)} S_l(a, \overline{a}) > 0. \tag{2.8}$$

Remark: The conditions (2.4), (2.6) and (2.7) imply that $S_l(F^p P_{w+l}, F^{w+l+1-p} P_{w+l}) = 0$. This condition and condition (2.8) say the pure Hodge structure $F^\bullet P_{w+l}$ of weight $w+l$ on P_{w+l} is polarized by S_l.

Let us fix one reference polarized Hodge structure $(H, H_{\mathbb{R}}, S, F_0^\bullet)$ of weight w. The space

$$\check{D} := \left\{ \text{filtrations } F^\bullet \text{ on } H \mid \dim F^p = \dim F_0^p,\ S(F^p, F^{w+1-p}) = 0 \right\}$$

is a closed submanifold of a product of Grassmannians, in particular projective. It is also a complex homogeneous space. Consider the subspace

$$D := \left\{ F^\bullet \in \check{D} \mid F^\bullet \text{ gives rise to a PHS, i.e., satisfies (2.1) and (2.3)} \right\}$$

which is an open complex submanifold and a real homogeneous space [Sch73]. It classifies polarized Hodge structures with fixed Hodge numbers.

Definition 2.4. *A tuple $(H, H_{\mathbb{R}}, S, F^\bullet, N)$ is said to give rise to a nilpotent orbit if the following holds:*

1. *$F^\bullet \in \check{D}$,*
2. *the endomorphism N of $H_{\mathbb{R}}$ is nilpotent and an infinitesimal isometry with $N(F^p) \subset F^{p-1}$,*
3. *there exists a bound $b \in \mathbb{R}$ such that*
$$e^{\rho N} F^\bullet \in D \text{ for } \Im(\rho) > b.$$

Then the set $\{e^{\rho N} F^\bullet \mid \rho \in \mathbb{C}\}$ is called a nilpotent orbit of Hodge structures.

Nilpotent orbits of Hodge structures play a fundamental role in Schmid's work [Sch73]. The following theorem gives a beautiful correspondence between PMHS and nilpotent orbits of Hodge structures. The main purpose of the whole paper is to generalize this correspondence to TERP-structures.

Theorem 2.5. *[Sch73][CK82][CKS86][CK89] Let $(H, H_{\mathbb{R}}, S)$ be as above.*

1. *The tuple $(H, H_{\mathbb{R}}, S, F^\bullet, N)$ is a PMHS of weight w if and only if $(H, H_{\mathbb{R}}, F^\bullet, N)$ gives rise to a nilpotent orbit of Hodge structures.*
2. *If $(H, H_{\mathbb{R}}, S, F^\bullet, N)$ is a PMHS with $I^{q,p} = \overline{I^{p,q}}$ then $e^{\rho N} F^\bullet \in D$ for $\Im(\rho) > 0$.*

The direction '$\Longleftarrow$' in *1.* is shown in [Sch73, Theorem 16.6.]. It is a consequence of the SL_2-orbit theorem. '$\Longrightarrow$' in *1.* is [CKS86, Corollary 3.13]. Short proofs of both directions are given in [CK89, Theorem 3.13]. The special case *2.* is proved in [CK82, Proposition 2.18] and in [CKS86, Lemma 3.12].
The nilpotent orbit theorem [Sch73] says that any variation of PHS on a punctured disk is approximated by a nilpotent orbit. Therefore the correspondence above associates a (limit) PMHS to any variation of PHS on a punctured disk.

3 TERP-structures and twistor structures

This chapter introduces the central objects of this paper: TERP-structures. This notion encapsulates a situation encountered when studying Hodge theory for singularities. More precisely, a TERP-structure arises when performing a Fourier-Laplace transformation of the Gauss-Manin-system and the Brieskorn lattice of a holomorphic function germ or a tame polynomial. In a sense which will become clear later (chapter 6), (variations of) TERP-structures are natural generalizations of (variations of) Hodge structures. We will give in this chapter the definitions and some properties of TERP-structures. The main point is the construction of a bundle on $\mathbb{P}^1$ starting from a given TERP-structure. After recalling the notion of (polarized) twistor structure, we will see that this $\mathbb{P}^1$-bundle is a (polarized) twistor with some additional structure, called integrable twistor. We only make some comments on how to extend these constructions to the case with parameters, i.e., for variations of TERP/twistor structures. In a sense, this chapter is a short version of the second chapter of [Her03] with some additional notations and comparison results.

Definition 3.1. *A TERP-structure ("twistor, extension, real structure, pairing") of weight $w \in \mathbb{Z}$ is a tuple $(H, H'_\mathbb{R}, \nabla, P, w)$ where H is a holomorphic vector bundle on $\mathbb{C}$, equipped with a flat meromorphic connection ∇ with a pole of order at most two at zero, a flat real subbundle $H'_\mathbb{R} \subset H' := H_{|\mathbb{C}^*}$ of the restriction to $\mathbb{C}^*$ satisfying $H' = H'_\mathbb{R} \otimes \mathbb{C}$ and a flat, bilinear, $(-1)^w$-symmetric, nondegenerate pairing*

$$P : H_z \times H_{-z} \longrightarrow \mathbb{C} \quad \text{for } z \in \mathbb{C}^*$$

with the following two properties.

1. *For any $z \in \mathbb{C}^*$, we have*
$$P : (H'_\mathbb{R})_z \times (H'_\mathbb{R})_{-z} \to i^w\mathbb{R}. \tag{3.1}$$

2. *The pairing induced on sections satisfies*
$$P : \mathcal{H} \otimes j^*\mathcal{H} \longrightarrow z^w\mathcal{O}_\mathbb{C}, \tag{3.2}$$
and the pairing $z^{-w}P$ is nondegenerate at 0.

Definition 3.2 (extension to infinity)**.** *Consider a TERP-structure $(H, H'_\mathbb{R}, \nabla, P, w)$. Let $\gamma : \mathbb{P}^1 \to \mathbb{P}^1; z \mapsto \overline{z}^{-1}$ and define for any $z \in \mathbb{C}^*$ the following two anti-linear involutions.*

$$\begin{array}{rcl} \tau_{real} : H_z & \longrightarrow & H_{\gamma(z)} \\ s & \longmapsto & \nabla\text{-parallel transport of } \overline{s} \end{array}$$

$$\begin{array}{rcl} \tau : H_z & \longrightarrow & H_{\gamma(z)} \\ s & \longmapsto & \nabla\text{-parallel transport of } \overline{z^{-w}s} \end{array}$$

The induced maps on sections by putting $s \mapsto \big(z \mapsto \tau s(\overline{z}^{-1})\big)$ resp. $s \mapsto \big(z \mapsto \tau_{real} s(\overline{z}^{-1})\big)$ will be denoted by the same letter. They can either be seen as morphisms $\tau, \tau_{real} : \mathcal{H}' \to \overline{\gamma^*\mathcal{H}'}$ which fix the base, or as morphisms $\tau, \tau_{real} : \mathcal{H}' \to \mathcal{H}'$ which map sections in $U \subset \mathbb{C}^*$ to sections in $\gamma(U) \subset \mathbb{C}^*$. Note that due to the two-fold conjugation (in the base and in fibres), τ and τ_{real} are morphisms of holomorphic bundles over $\mathbb{C}^*$. Denote by $\widehat{H} \in VB_{\mathbb{P}^1}$ the bundle obtained by patching $\mathcal{H}$ and $\overline{\gamma^*\mathcal{H}}$ via the identification τ.

Notice that the pairing P does not enter in the construction of the bundle $\widehat{H}$. However, the sole fact that the bundle H is equipped with a pairing with the above properties puts restrictions on $\widehat{H}$ as the following lemma shows. Let us denote the sheaf $\mathcal{O}(\widehat{H}_{|\mathbb{P}^1\setminus\{0\}})$ by $\widetilde{\mathcal{H}}$ for short.

Lemma 3.3.

1. *The connection naturally extends with a pole of order two at infinity.*

2. *The pairing P satisfies $P : \widetilde{\mathcal{H}} \otimes j^*\widetilde{\mathcal{H}} \to z^w\mathcal{O}_{\mathbb{P}^1\setminus\{0\}}$, and $z^{-w}P$ is nondegenerate at ∞.*

3. *The bundle $\widehat{H}$ has degree zero.*

Proof. We will need the following two equalities of endomorphisms of $\mathcal{H}'$, which express the flatness property of τ_{real}. They are immediate consequences of $\overline{\gamma}^*\left(\frac{dz}{z}\right) = -\frac{dz}{z}$.

$$\nabla_{z\partial_z} \circ \tau_{real} = \tau_{real} \circ \nabla_{-z\partial_z} \quad ; \quad \nabla_{z\partial_z} \circ \tau = \tau \circ (\nabla_{-z\partial_z} + wId)\,. \tag{3.3}$$

Consider $\mathcal{H}$ as a subsheaf of $i_*\mathcal{H}'$ where $i : \mathbb{C}^* \hookrightarrow \mathbb{C}$. By definition, we have $\widetilde{\mathcal{H}} = \tau\mathcal{H} \subset \widetilde{i}_*\mathcal{H}$ with $\widetilde{i} : \mathbb{C}^* \hookrightarrow \mathbb{P}^1\backslash\{0\}$. This gives immediately, using the above formula, that

$$z^{-1}\nabla_{\partial_{z^{-1}}}\widetilde{\mathcal{H}} = -\nabla_{z\partial_z}\tau\mathcal{H} = \tau(\nabla_{z\partial_z} - wId)\mathcal{H} = \tau(\frac{1}{z}\mathcal{H}) = z\widetilde{\mathcal{H}}.$$

To detect the order of P at infinity, consider the following calculation:

$$\begin{aligned}
\overline{z^{-w} \cdot P(a(z), b(-z))} &= (-\overline{z})^w \cdot \overline{P(z^{-w}a(z), (-z)^{-w}b(-z))} \\
&= \overline{z}^w \cdot P(\overline{z^{-w}a(z)}, \overline{(-z)^{-w}b(-z)}) \qquad [(-1)^w \text{ because of condition (3.1)}] \\
&= \overline{z}^w \cdot P\left(\tau(a)(\overline{z}^{-1}), \tau(b)(-\overline{z}^{-1})\right) \qquad [P \text{ is flat}].
\end{aligned} \tag{3.4}$$

The order of P at zero and $\widetilde{\mathcal{H}} = \tau\mathcal{H}$ yield $P : \widetilde{\mathcal{H}} \otimes j^*\widetilde{\mathcal{H}} \to z^w\mathcal{O}_{\mathbb{P}^1\backslash\{0\}}$ as required. To prove the third point, we first consider the case where $rank(H) = 1$. Then $\widehat{\mathcal{H}} \cong \mathcal{O}_{\mathbb{P}^1}(k)$ for some $k \in \mathbb{Z}$. Choose non-vanishing holomorphic sections $\sigma \in \Gamma(\mathbb{C}, \widehat{\mathcal{H}})$ and $\widetilde{\sigma} \in \Gamma(\mathbb{P}^1\backslash\{0\}, \widehat{\mathcal{H}})$ satisfying $\widetilde{\sigma} = z^k\sigma$ on $\mathbb{C}^*$. This implies that $z \mapsto P(\sigma(z), \sigma(-z))$ defines a non-vanishing holomorphic function on $\mathbb{C}^*$ with a zero of order w at zero (by equation (3.2) in the definition of TERP-structures) and a zero of order $2k - w$ at infinity by the above computation. Consequently, $k = 0$. Now for the general case, we remark that given any TERP-structure $(H, H'_{\mathbb{R}}, \nabla, P, w)$, then the determinant (line) bundle $\det(H)$ is naturally a TERP-structure of weight $w{\cdot}rank(H)$. Moreover, it is clear that $\widehat{\det(H)} = \det(\widehat{H})$. This implies that $\deg(\widehat{H}) = 0$. $\square$

The next step is to investigate more closely the case where $\widehat{H}$ is a trivial $\mathbb{P}^1$-bundle. This implies that we have a canonical identification of any fibre with the space $H^0(\mathbb{P}^1, \widehat{\mathcal{H}})$ and thus also a canonical identification of all fibres.

Lemma 3.4. *Let $\widehat{H}$ be trivial and consider the identification $H_0 \xrightarrow{\cong} H^0(\mathbb{P}^1, \widehat{\mathcal{H}})$. The morphism τ acts on this space as an anti-linear involution and the pairing $z^{-w}P$ is symmetric and has constant values on it. Define*

$$h : H_0 \times H_0 \longrightarrow \mathbb{C} \quad ; \quad (a, b) \longmapsto z^{-w}P(a, \tau b).$$

Then h is a hermitian pairing on H_0.

Proof. In order to see that τ defines an anti-holomorphic involution on $H^0(\mathbb{P}^1, \widehat{\mathcal{H}})$, consider the extension $\widehat{i}_*\mathcal{H}'$, where $\widehat{i} : \mathbb{C}^* \hookrightarrow \mathbb{P}^1$. We have $\widetilde{i}_*\mathcal{H} \subset \widehat{i}_*\mathcal{H}'$, $i_*\widetilde{\mathcal{H}} \subset \widehat{i}_*\mathcal{H}'$, and $H^0(\mathbb{P}^1, \widetilde{i}_*\mathcal{H} \cap i_*\widetilde{\mathcal{H}})$ is precisely the finite-dimensional space $H^0(\mathbb{P}^1, \widehat{\mathcal{H}})$ which is of dimension equal to the rank of H if $\widehat{H}$ is trivial. The morphism τ acts on $\mathcal{H}'$ and therefore on $\widehat{i}_*\mathcal{H}'$. It maps $\widetilde{i}_*\mathcal{H}$ isomorphically to $i_*\widetilde{\mathcal{H}}$ and vice versa. This shows that it acts on $H^0(\mathbb{P}^1, \widehat{\mathcal{H}})$. For any two global sections $a, b \in H^0(\mathbb{P}^1, \widehat{\mathcal{H}})$, putting $z \mapsto P\left(a(z), b(-z)\right)$ defines a holomorphic function with zero of order w at the origin and pole of order w at infinity. Therefore, the function $z \mapsto z^{-w}P\left(a(z), b(-z)\right)$ is holomorphic on $\mathbb{P}^1$ and thus constant. The symmetry property follows from

$$z^{-w}P\left(a, b\right)(z) = z^{-w}P\left(a(z), b(-z)\right) = z^{-w}(-1)^w P\left(b(-z), a(z)\right) = z^{-w}P\left(b, a\right)(z).$$

In order to show that h is hermitian, we apply computation (3.4) to global sections, which gives $\overline{z^{-w}P(a, b)} = z^{-w}P(\tau a, \tau b)$ for $a, b \in H^0(\mathbb{P}^1, \widehat{\mathcal{H}})$. This implies that

$$h(a, b) = z^{-w}P(a, \tau b) = \overline{z^{-w}P(\tau a, b)} = \overline{z^{-w}P(b, \tau a)} = \overline{h(b, a)}$$

which is what we need. $\square$

The last lemma motivates the following definition.

Definition 3.5. *A TERP-structure is called pure iff the bundle $\widehat{H}$ is trivial. A pure TERP-structure is called polarized iff the hermitian form $h : H_0 \times H_0 \to \mathbb{C}$ is positive definite.*

The next result introduces one of the most interesting objects attached to a pure TERP-structure, namely, an endomorphism of H_0 that was considered in [CFIV92] under the name "new supersymmetric index". Its eigenvalues are related to and can be considered (in the regular singular case, see chapter 6) as a generalization of the spectral numbers of H, ∇. Let $\mathcal{U}$ be the pole part of the connection ∇ on H, i.e. $\mathcal{U}$ is an endomorphism of the fibre H_0 defined by $\mathcal{U} = [z\nabla_{z\partial_z}]$.

Lemma 3.6. *Suppose that $(H, H'_\mathbb{R}, \nabla, P, w)$ is a pure TERP-structure. Then there exists an endomorphism $\mathcal{Q}$ of H_0 such that for any $\omega \in H^0(\mathbb{P}^1, \widehat{\mathcal{H}}) \cong H_0$, we have*

$$\nabla_{z\partial_z}\omega = \left(\frac{1}{z}\mathcal{U} + \left(\frac{w}{2}Id - \mathcal{Q}\right) - z\overline{\mathcal{U}}\right)\omega$$

where $\overline{\mathcal{U}}$ denotes the adjoint of $\mathcal{U}$ with respect to h and satisfies $\overline{U} = \tau \circ U \circ \tau$. $\mathcal{Q}$ is h-selfadjoint and anti-commutes with τ. If the TERP-structure is polarized, $\mathcal{Q}$ is semi-simple and its eigenvalues are real and symmetric with respect to zero.

Proof. It is obvious that there are endomorphisms A, B such that $\nabla_{z\partial_z}\omega = \left(\frac{\mathcal{U}}{z} + A + zB\right)\omega$. We need to show that $B = -\overline{\mathcal{U}}$, that $\mathcal{Q} := \frac{w}{2}Id - A$ is h-selfadjoint and anti-commutes with τ and that $\overline{U} = \tau \circ U \circ \tau$. All of these properties follow using the equations (3.3), $\tau(z\omega) = z^{-1}\tau(\omega)$ and the fact that for $a, b \in H^0(\mathbb{P}^1, \widehat{\mathcal{H}})$ we have $z^{-w}P(\mathcal{U}a, b) = z^{-w}P(a, \mathcal{U}b)$. If h is positive definite, $\mathcal{Q}$ is semi-simple with real eigenvalues. They are symmetric because of $\tau \circ \mathcal{Q} = -\mathcal{Q} \circ \tau$. $\square$

We will give in the following definition/theorem a brief reminder on how to extend the notion of a TERP-structure to the relative case, where parameters have to be taken into account. The main reference is [Her03, chapter 2], in particular sections 2.4 to 2.7 of cit.loc.

Definition-Theorem 3.7. *Let M be a complex manifold. A variation of TERP-structures over M is a tuple $(H, H'_\mathbb{R}, \nabla, P, w)$ where $H \in VB_{\mathbb{C}\times M}$ and $H'_\mathbb{R}$ a maximal real subbundle of the restriction $H' := H_{|\mathbb{C}^*\times M}$. The connection $\nabla : \mathcal{H}' \to \mathcal{H}' \otimes \Omega^1_{\mathbb{C}\times M}$ is flat and meromorphic with a pole of Poincar rank one along $\{0\} \times M$. The pairing $P : \mathcal{H} \otimes j^*\mathcal{H} \to z^w\mathcal{O}_{\mathbb{C}\times M}$ is non-degenerate, $(-1)^w$-symmetric, flat and sends $H'_\mathbb{R}$ to $i^w\mathbb{R}$.*
The following facts hold:

1. *The construction of the extension to infinity generalizes and yields a complex vector bundle $\widehat{H}$ over $\mathbb{P}^1 \times M$ with holomorphic structure in $\mathbb{P}^1$-direction, in other words, a locally free $\mathcal{C}^\infty_M\mathcal{O}_{\mathbb{P}^1}$-module. The connection extends to $\widehat{H}$ with a pole of Poincar rank one along $\{0, \infty\} \times M$.*

2. *The notion of pure resp. polarized pure TERP-structures is defined as in the absolute case. Given a variation of $(H, H'_\mathbb{R}, \nabla, P, w)$ of pure TERP-structures, the objects $h, \tau, \mathcal{U}, Q$ are defined on $p_*\mathcal{C}^{\infty h}(\widehat{H}) \cong \mathcal{O}(H_{|z=0}) \otimes_{\mathcal{O}_M} \mathcal{C}^\infty_M$. Moreover precisely, the connection ∇ takes the following form: Let $\omega \in p_*\mathcal{C}^{\infty h}(\widehat{H})$, then*
$$\nabla\omega = \left(D + \frac{1}{z}C + z\overline{C} + \left(\frac{1}{z}\mathcal{U} + \left(\frac{w}{2}Id - \mathcal{Q}\right) - z\overline{\mathcal{U}}\right)\frac{\mathrm{d}z}{z}\right)\omega$$
where D is the Chern connection for h, C a Higgs field on $H_{|z=0}$ and $\overline{C}$ its h-adjoint. The operators $D, C, \overline{C}, h, \tau, \mathcal{U}, Q$ satisfy a couple of compatibility conditions ([Her03, equations 2.50-2.61]) making up what was called CV-structure in cit.loc.

In the remaining part of this chapter, we will discuss the relation of the notion of (variation of) TERP-structures with (variation) of polarized integrable twistor structures. Polarized twistor structures were defined in [Sim97], and the term integrable was first used in [Sab05b]. We briefly recall the definitions.
As in [Sim97], we denote for any $\mathcal{O}_{\mathbb{P}^1}$-module $\mathcal{E}$ by $\sigma^*\mathcal{E}$ the sheaf defined by $\Gamma(U, \sigma^*\mathcal{E}) := \overline{\Gamma(\sigma(U), \mathcal{E})}$. As before, the conjugate complex structure is needed to ensure that $\sigma^*\mathcal{E}$ is again a sheaf of $\mathcal{O}_{\mathbb{P}^1}$-modules. Note that the convention here differs from the one used for the map γ, but we prefer to be compatible with the notations both in [Her03] and [Sim97].

Definition 3.8.
- *A twistor is a holomorphic bundle on $\mathbb{P}^1$.*
- *A twistor $\widehat{H} \in VB_{\mathbb{P}^1}$ is integrable if it comes equipped with a meromorphic connection with poles of order at most two at zero and infinity.*
- *$\widehat{H}$ is called pure of weight w iff it is semi-stable of slope w, i.e., isomorphic to a sum $\oplus_{i=1}^{rank(\widehat{H})}\mathcal{O}_{\mathbb{P}^1}(w)$.*
- *A pairing on a twistor $\widehat{H}$ is a non-degenerate $(-1)^k$-symmetric morphism $\widehat{S} : \mathcal{O}(\widehat{H}) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \sigma^*\mathcal{O}(\widehat{H}) \to \mathcal{O}_{\mathbb{P}^1}(2k)$. In case $\widehat{H}$ is integrable $\widehat{S}$ is required to be flat.*
- *A pure twistor $\widehat{H}$ of weight w is called polarized by a pairing $\widehat{S}$ iff the induced pairing*
$$\widehat{S}_w : \Big(\mathcal{O}(\widehat{H}) \otimes \mathcal{O}_{\mathbb{P}^1}(-w)\Big) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \sigma^* \Big(\mathcal{O}(\widehat{H}) \otimes \mathcal{O}_{\mathbb{P}^1}(-w)\Big) \longrightarrow \mathcal{O}_{\mathbb{P}^1}$$
induces a positive definite hermitian pairing on the space of global sections. The pair $(\widehat{H}, \widehat{S})$ is called polarized twistor structure (abbreviation PTS).
- *A twistor $\widehat{H}$ is mixed iff it is equipped with an increasing filtration $\widehat{W}_\bullet$ by subbundles such that each graded piece $Gr_k^{\widehat{W}}(\widehat{H})$ is a pure twistor of weight k.*

With these definitions in mind, we can state the following comparison lemma.

Lemma 3.9. *Let $(H, \nabla, H_{\mathbb{R}}, P, w)$ be a TERP-structure. Then $(\widehat{H}, \nabla)$ is an integrable twistor. There is a naturally defined pairing $\widehat{S} : \mathcal{O}(\widehat{H}) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \sigma^*\mathcal{O}(\widehat{H}) \to \mathcal{O}_{\mathbb{P}^1}$. $\widehat{H}$ is pure (and then automatically of weight zero) iff H is pure TERP and in that case $\widehat{S}$ gives a polarization iff H is polarized pure TERP.*

Proof. The statements about $(\widehat{H}, \nabla)$ are obvious from what has been said before, it was shown in lemma 3.3 that the connection extends to $\widehat{H}$ as required and that $\deg(\widehat{H}) = 0$. The pairing $\widehat{S}$ is defined as
$$\widehat{S}(a, b) := (-1)^w P(a, \tau_{real} b) = z^{-w} P(a, \tau b).$$
P has by definition a zero of order w at the origin and (as was shown) a pole of order w at infinity, which implies that $\widehat{S}$ maps to $\mathcal{O}_{\mathbb{P}^1}$. The flatness of $\widehat{S}$ follows from the flatness of P and τ_{real}. The only thing that remains to discuss is that if $\widehat{H}$ is pure then it is polarized by $\widehat{S}$ precisely iff the TERP-structure we started with is polarized pure TERP. But this is a tautology: $\widehat{S}$ polarizes $\widehat{H}$ iff the induced form on $H^0(\mathbb{P}^1, \mathcal{O}(\widehat{H}))$ is positive definite hermitian, but this form is exactly $z^{-w}P(-, \tau(-))$. The positive definiteness of this form on the space of global sections was the defining property for a pure TERP-structure to be polarized pure TERP. ☐

Remark: A TERP-structure $(H, \nabla, H'_{\mathbb{R}}, P)$ comes equipped with an integer w, its weight. However, lemma 3.3 shows that the twistor $\widehat{H}$ constructed from such a TERP-structure is a $\mathbb{P}^1$-bundle which always has degree zero. Consequently, for a pure TERP-structure, the twistor $\widehat{H}$ is pure of weight zero, regardless of the value of w. The reason for this is that the Tate twist, which is used to transform a pure twistor of some weight in a twistor of weight zero is already implicitly contained in our gluing constructing from definition 3.2, namely, $\widehat{H}$ is defined by patching H and $\overline{\gamma^* H}$ via the map τ, and not via τ_{real}.

In [Sab05b], the related notion of an $\mathcal{R}$-triple was introduced. The following lemma gives the comparison. We omit the proof, which is more or less straightforward.

Lemma 3.10. *Given a TERP-structure $(H, \nabla, H'_{\mathbb{R}}, P, w)$, the tuple $(\mathcal{O}(H), \mathcal{O}(H), \widehat{S})$ where $\widehat{S}$ is the above pairing restricted to $\mathcal{O}(H')$ is a smooth $\mathcal{R}$-triple. It is equal to its hermitian adjoint $(\mathcal{O}(H), \mathcal{O}(H), \widehat{S}^*)$, so it is polarized by (Id, Id). It is even an object in $\mathcal{R}int(pt)$, the vertical connection being ∇_z (and using the flatness of $\widehat{S}$).*

As before, we need a relative version of the above notions taking parameters into account. It can be formulated in two equivalent ways.

Definition 3.11 (Variation of twistor structures, harmonic bundles)**.** *Let M be a complex manifold.*

1. *Consider a $\mathcal{C}^\infty$ vector bundle on $\mathbb{P}^1 \times M$ together with an integrable operator $\overline{\partial}_{\mathbb{P}^1} : \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(E) \otimes_{\mathcal{C}^\infty_{M\times\mathbb{P}^1}} \mathcal{C}^\infty_M \mathcal{A}^{0,1}_{\mathbb{P}^1}$ defining a locally free sheaf $\mathcal{E}$ of $\mathcal{C}^\infty_M \mathcal{O}_{\mathbb{P}^1}$-modules. Then $\mathcal{E}$ is called a variation of twistor structures (abbreviation VTS) if it comes equipped with an $\mathcal{O}_{\mathbb{P}^1}$-linear operator $\mathbb{D} : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{C}^\infty_M \mathcal{O}_{\mathbb{P}^1}} \xi\mathcal{A}^1_M$ satisfying the following Leibniz rule:*
$$\mathbb{D}(fe) = f\mathbb{D}(e) + \mathbf{d}(f)e$$
where $\xi\mathcal{A}^1$ is the twistor $\mathcal{A}^1_M \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1}(1)$ (see [Sim97] or [Moc07]) and $\mathbf{d} : \xi\mathcal{C}^\infty_M \cong \mathcal{O}_{\mathbb{P}^1} \otimes_{\mathbb{C}} \mathcal{C}^\infty_M \to \xi\mathcal{A}^1_M$ is the natural "twistor derivative". A variation of polarized twistor structures (abbreviation VPTS) is a variation equipped with a polarization as in definition 3.8 which is flat with respect to $\mathbb{D}$. Similarly, one defines variations of pure resp. mixed twistor structures.

2. *A $\mathcal{C}^\infty$-bundle E on M is called harmonic iff there are operators $\partial, \overline{\partial}, \theta, \overline{\theta}$ and a pairing h with*
$$\begin{array}{rcccl} \partial, \theta & : & \mathcal{C}^\infty(E) & \longrightarrow & \mathcal{C}^\infty(E) \otimes_{\mathcal{C}^\infty_M} \mathcal{A}^{1,0}_M \\ \overline{\partial}, \overline{\theta} & : & \mathcal{C}^\infty(E) & \longrightarrow & \mathcal{C}^\infty(E) \otimes_{\mathcal{C}^\infty_M} \mathcal{A}^{0,1}_M \\ h & : & \mathcal{C}^\infty(E) \otimes_{\mathcal{C}^\infty_M} \mathcal{C}^\infty(\overline{E}) & \longrightarrow & \mathcal{C}^\infty_M \end{array}$$
where $\theta, \overline{\theta}$ are $\mathcal{C}^\infty_M$-linear and $\partial, \overline{\partial}$ are satisfying the Leibniz-rule, such that $(\partial + \overline{\partial} + \theta + \overline{\theta})^2 = 0$. The pairing h is positive definite, $\overline{\theta} + \theta$ is h-self-adjoint and $(\partial + \overline{\partial})$ is h-metric, i.e. $dh(a,b) = h(\partial a, b) + h(a, \overline{\partial} b)$ and $\overline{d}h(a,b) = h(\overline{\partial} a, b) + h(a, \partial b)$.

A basic result due to Simpson ([Sim97]) is that the category of variations of pure polarized twistor structures of weight zero is equivalent to the category of harmonic bundles. This correspondence will be used implicitly several times in the sequel. As a matter of notation, for a variation of twistors $\widehat{E}$ on M, we denote by $\mathcal{C}^{\infty h}(\widehat{E})$ the sheaf of $\mathcal{C}^\infty$-sections of $\widehat{E}$ which are holomorphic in the $\mathbb{P}^1$-direction, i.e., annihilated by the operator $\overline{\partial}_{\mathbb{P}^1}$. The following extension of lemma 3.9 to the relative case is a condensed version of [Her03, theorem 2.19] (see also [Sab05b, corollary 7.2.6]).

Lemma 3.12. *For any variation of polarized pure TERP-structures $(G, G'_{|\mathbb{C}^*\times M}, \nabla, P, w)$, $\widehat{G}$ has the structure of a variation of pure polarized integrable twistors structures of weight zero yielding a harmonic bundle $p_*\mathcal{C}^{\infty h}(\widehat{G})$ on M equipped with operators $\mathcal{U}, \mathcal{Q}$ satisfying [Sab05b, equations 7.2.5].*

Proof. We only remark how to define the connection in the parameter direction: By definition, a variation of TERP-structures furnishes a "horizontal" connection $\nabla_M : \mathcal{O}(G) \to \mathcal{O}(G) \otimes z^{-1}\mathcal{O}_{\mathbb{C}}\Omega^1_M$ which we can of course extend to
$$D'_M : \mathcal{C}^{\infty h}(G) \longrightarrow \mathcal{C}^{\infty h}(G) \otimes \left(z^{-1}\mathcal{O}_{\mathbb{C}}\mathcal{C}^\infty_M\mathcal{A}^{1,0}_M \oplus \mathcal{O}_{\mathbb{C}}\mathcal{C}^\infty_M\mathcal{A}^{0,1}_M\right)$$
Similarly, the morphism $\gamma^*\nabla_M$ induces the operator
$$D'' : \mathcal{C}^{\infty h}(\overline{\gamma^* G}) \longrightarrow \mathcal{C}^{\infty h}(\overline{\gamma^* G}) \otimes \left(z\mathcal{O}_{\mathbb{P}^1\backslash\{0\}}\mathcal{C}^\infty_M\mathcal{A}^{0,1}_M \oplus \mathcal{O}_{\mathbb{P}^1\backslash\{0\}}\mathcal{C}^\infty_M\mathcal{A}^{1,0}\right)$$
Then we put $\mathbb{D} : \mathcal{C}^{\infty h}(\widehat{G}) \to \mathcal{C}^{\infty h}(\widehat{G}) \otimes \xi\mathcal{A}^1_M$ by $\mathbb{D} = zD' + z^{-1}D''$ where z and z^{-1} are seen as global sections of $\mathcal{O}_{\mathbb{P}^1}(1)$. □

Remark: It should be more or less clear from what has been said that the notion of integrable twistor also extends to the relative case, namely, given a variation of twistor structures $(\mathcal{E}, \mathbb{D})$ like before then we call it integrable if on both charts $\mathbb{C} \times M$ and $(\mathbb{P}^1\backslash\{0\}) \times \overline{M}$ the relative meromorphic connections defined by $\mathbb{D}$ can be completed to an absolute meromorphic connection having poles of Poincar rank one at $\{0\} \times M$ and $\{\infty\} \times \overline{M}$. This also explains the term integrable. Each restriction $\mathcal{E}_{\mathbb{P}^1\times\{x\}}$ for $x \in M$ is then naturally an integrable twistor as defined before. For more details, see [Sab05b, chapter 7].

4 Nilpotent orbits of TERP-structures

After the general discussion of TERP-structures and variations of them in the previous chapter, we will introduce now a particular class of such variations over one-dimensional bases, these are called nilpotent orbits. The name

is derived from corresponding objects in Hodge theory ([Sch73]), and it will become clear later (see lemma 6.5) that nilpotent orbits of TERP-structures give rise to nilpotent orbits of Hodge-structures in the classical sense. We will simultaneously consider two points of view: Starting with a single TERP-structure, there is a canonical way to construct variations over punctured discs, on the other hand, given such a variation, we will give a simple criterion to decide whether it is a nilpotent orbit.

Definition 4.1. *Consider the following holomorphic maps: for any $r \in \mathbb{C}^*$, let $\pi_r : \mathbb{C} \to \mathbb{C}$ be the multiplication by r and define $\pi, \pi' : \mathbb{C} \times \mathbb{C}^* \to \mathbb{C}$ by $\pi(z,r) = zr$ and $\pi'(z,r) = zr^{-1}$.*

1. *Let $(H, H'_\mathbb{R}, \nabla, P, w)$ be a TERP-structure. Then $\pi_r^*(H, H'_\mathbb{R}, \nabla, P, w)$ is also TERP, and we say that $(H, H'_\mathbb{R}, \nabla, P, w)$ induces a nilpotent orbit iff $\pi_r^*(H, H'_\mathbb{R}, \nabla, P, w)$ is a polarized pure TERP-structure for any $r \in \mathbb{C}^*$ with $|r| \ll 1$. Similarly, $(H, H'_\mathbb{R}, \nabla, P, w)$ is said to induce a Sabbah orbit iff $\pi_r^*(H, H'_\mathbb{R}, \nabla, P, w)$ is polarized pure TERP for $|r| \gg 0$, i.e. iff $\pi_{r^{-1}}^*(H, H'_\mathbb{R}, \nabla, P, w)$ is polarized pure TERP for $|r| \ll 1$*
2. *Let $M = \Delta^*$ be a punctured unit disk with coordinate r, and $(G, G'_\mathbb{R}, \nabla, P, w)$ be a variation of TERP-structures on M (it might even be defined on a larger disc or on the whole of $\mathbb{C}^*$). Then we call it nilpotent orbit (resp. Sabbah orbit) iff*
   - *(a) The sheaf $\mathcal{O}(G)$ is stable under $\nabla_{z\partial_z - r\partial_r}$ (resp. under $\nabla_{z\partial_z + r\partial_r}$).*
   - *(b) For any $r \in M$ with $|r| \ll 1$, the restriction $(G, \nabla, H'_\mathbb{R}, P, w)_{|\mathbb{C}\times\{r\}}$ is a polarized pure TERP-structure.*

We will show that if a TERP-structure induces a nilpotent orbit resp. a Sabbah orbit, then the family $\pi^*(H, \nabla, H'_\mathbb{R}, P, w)$ (resp. $(\pi')^*(H, \nabla, H'_\mathbb{R}, P, w)$) is a nilpotent orbit resp. Sabbah orbit according to the second definition above and that vice versa, any nilpotent resp. Sabbah orbit is of this type. For this purpose and also for later use, we will discuss the different types of monodromy involved in this situation. For any flat bundle on $\mathbb{C}^* \times \Delta^*$, we call the monodromy of the loop $(z_0 e^{i\varphi}, r_0)$ vertical, and that of $(z_0, r_0 e^{i\varphi})$ horizontal (here $z_0, r_0 \neq 0$ and $\varphi \in [0, 2\pi)$). We will mainly treat nilpotent orbits, and only comment on the case of Sabbah orbits which is quite analogous.

Lemma 4.2. *Suppose that $(H, \nabla, H'_\mathbb{R}, P, w)$ induces a nilpotent orbit. Let $G := \pi^* H$. Consider the restriction of $G_{|\pi^{-1}(z_0)}$ to a fibre of π (which is isomorphic to $\mathbb{C}^*$). This bundle is canonically trivialized and we denote by $\rho_{r,z_0} : G_{(z_0 r^{-1}, r)} \to G_{(z_0,1)}$ the identification of the fibres of G. For $z_0 \in \mathbb{C}^*$ this trivialization is given by the flat structure of $G_{|\pi^{-1}(z_0)}$. In particular, the monodromy of $G_{|\pi^{-1}(z_0)}$ is trivial. This implies that vertical and horizontal monodromy of $G_{|\mathbb{C}^*\times M}$ coincide.*

Proof. The first two statements are clear, they follow from the definition of the pull-back of a bundle with flat connection (see, e.g., [Sab02]). The monodromy of a flat bundle on $\pi^{-1}(z_0) \cong \mathbb{C}^*$ which has a basis of flat sections is obviously trivial. Moreover, a counter-clockwise oriented loop inside a fibre $\pi^{-1}(z_0)$ is homotopic to the composition of $(z_0 e^{i\varphi}, r_0)$ and $(z_0, r_0 e^{-i\varphi})$, so that vertical and horizontal monodromy must be equal. □

Lemma 4.3. *A TERP-structure $(H, \nabla, H'_\mathbb{R}, P, w)$ induces a nilpotent orbit (resp. a Sabbah orbit) iff the variation $\pi^*(H, \nabla, H'_\mathbb{R}, P, w)$ (resp. $\pi'^*(H, \nabla, H'_\mathbb{R}, P, w)$) is a nilpotent orbit (resp. a Sabbah orbit). Any nilpotent orbit (resp. Sabbah orbit) on a punctured disk containing 1 is induced from its restriction to $\mathbb{C} \times \{1\}$ by the map π (resp. π').*

Proof. Suppose that $(H, \nabla, H'_\mathbb{R}, P, w)$ induces a nilpotent orbit. The restriction of $\pi^*(H, \nabla, H'_\mathbb{R}, P, w)$ to $\mathbb{C} \times \{r\}$ is equal to $\pi_r^*(H, \nabla, H'_\mathbb{R}, P, w)$ by definition and therefore a polarized pure TERP-structure. We need to show that the family $G := \pi^* H$ is a variation of TERP-structures, i.e, that the connection has a pole of Poincar rank at most one along $\{0\} \times \Delta^*$ and that $\mathcal{O}(\pi^* H)$ is stable under $\nabla_{z\partial_z - r\partial_r}$. It is readily checked that the fibres of π are precisely the integral curves of the vector field $z\partial_z - r\partial_r$. Lemma 4.2 implies that for any $\sigma \in \mathcal{O}(H)$, the pullback $\pi^*\sigma \in \pi^{-1}\mathcal{O}(H) \subset \mathcal{O}_{\mathbb{C}\times M} \otimes \pi^{-1}\mathcal{O}(H) = \mathcal{O}(\pi^* H)$ satisfies $\nabla_{z\partial_z - r\partial_r} \pi^*\sigma = 0$. By definition, $\mathcal{O}(\pi^* H)$ is generated by such sections $\pi^*\sigma$, therefore, $\mathcal{O}(\pi^* H)$ must be stable under $\nabla_{z\partial_z - r\partial_r}$. Moreover, it is clear that $\nabla_{z^2\partial_z} \mathcal{O}(\pi^* H) \subset \mathcal{O}(\pi^* H)$, as this is true for the restriction to any $\mathbb{C} \times \{r\}$. Finally, it follows from the identity $z\partial_r = \frac{1}{r} z^2 \partial_z - \frac{z}{r}(z\partial_z - r\partial_r)$ that $\mathcal{O}(\pi^* H)$ is also stable under $\nabla_{z\partial_r}$. In conclusion, we have shown that $\pi^*(H, \nabla, H'_\mathbb{R}, P, w)$ is a nilpotent orbit.

Conversely, given any bundle $G \in VB_{\mathbb{C}\times\Delta^*}$ which underlies a variation of polarized pure TERP-structures and satisfies $\nabla_{z\partial_z - r\partial_r} \mathcal{O}(G) \subset \mathcal{O}(G)$, we need to see that it is of the form $G = \pi^* G_{|\mathbb{C}\times\{1\}}$. This follows from the

calculation done in lemma 7.19 of [Her03], i.e., the fact that if $\nabla_{z\partial_z - r\partial_r}$ sends $\mathcal{O}(G)$ to itself then any base $\underline{e}$ of this sheaf is related to the base $\pi^*(\underline{e}_{|\mathbb{C}\times\{1\}})$ by a base change in $GL(\mathcal{O}_{\mathbb{C}\times\Delta^*})$. $\square$

In the following lemma, we consider the bundle $\widehat{H}$ over $\mathbb{P}^1$ obtained from a TERP-structure by the construction of definition 3.2. If we are given a variation of TERP-structures, then by [Her03], lemma 2.14 (e), one ends up with a real analytic family of holomorphic $\mathbb{P}^1$-bundles. In the case of a variation π^*H of the above type, the following statement shows that the canonical identification ρ_{r,z_0} of fibres of π^*H over points $(z,r) \in \pi^{-1}(z_0)$ extends to an identification of the $\mathbb{P}^1$-bundles $\widehat{\pi^*H}_{|\mathbb{P}^1\times\{r\}}$ for all $r \in S^1$ and more generally we have $\widehat{\pi^*H}_{|\mathbb{P}^1\times\{r\}} \cong \widehat{\pi^*H}_{|\mathbb{P}^1\times\{r'\}}$ if $|r| = |r'|$. This means that the "interesting" part of the variation $\pi^*H \to \mathbb{C}\times M$ is the restriction to rays in M with fixed argument.

Lemma 4.4. *For any $r \in S^1$, there is a canonical bundle isomorphism $\widehat{\pi_r^*H} \to \widehat{H}$ which restricts over any $z_0 \in \mathbb{C} \subset \mathbb{P}^1$ to the isomorphism of fibres $\rho_{r,z_0} : (\pi^*H)_{|(z_0\cdot r^{-1},r)} \to H_{|z_0}$ considered in lemma 4.2.*

Proof. The pointwise isomorphisms ρ_{r,z_0} glue to a bundle isomorphisms $\phi_r : \pi_r^*H \to H$. Moreover, for $r \in S^1$ we have $1/r = \overline{r}$, and we can also glue the maps $\rho_{r,\overline{z_0}^{-1}} : (\pi^*H)_{|\overline{(r\cdot z_0^{-1},r)}} \to H_{|\overline{z_0}^{-1}}$ for all $z_0 \in \mathbb{P}^1\backslash\{0\}$ to an isomorphism $\widetilde{\phi}_r : \overline{\gamma^*(\pi_r^*H)} \to \overline{\gamma^*H}$. If we twist $\widetilde{\phi}_r$ by multiplication with r^w, we obtain the following commuting diagram, which shows that $\widehat{\pi_r^*H} \cong \widehat{H}$.

$$
\begin{tikzcd}
\text{flat shift of } \overline{(z\tfrac{1}{r})^{-w}b} & \overline{\gamma^*\pi_r^*H} \arrow[r, "r^w\cdot\widetilde{\phi}_r", "\cong"'] & \overline{\gamma^*H} & \text{flat shift of } \overline{z^{-w}a} \\
b \arrow[u, mapsto] & \pi_r^*H \arrow[u, "\tau"] \arrow[r, "\cong", "\phi_r"'] & H \arrow[u, "\tau"'] & a \arrow[u, mapsto]
\end{tikzcd}
$$

$\square$

The lemma shows in particular that for any $r \in S^1$, $\widehat{\pi_r^*H}$ is a trivial bundle iff $\widehat{H}$ is so. We obtain as a consequence:

Corollary 4.5. *For any bundle $G \in VB_{\mathbb{C}\times\Delta^*}$ underlying a variation of TERP-structures with the property that $\nabla_{z\partial_z - r\partial_r}\mathcal{O}(G) \subset \mathcal{O}(G)$, the subset $\left\{r \in \Delta^* \mid \widehat{G}_{|\mathbb{P}^1\times\{r\}} \text{ is not trivial}\right\}$ is either the whole of Δ^* or a discrete union $\bigcup S_c$ with $S_c = \{r \in \mathbb{C}^* \mid |r| = c\}$ or empty. On the complement, one obtains a real analytic hermitian form h by the procedure of lemma 3.4. Its signature is constant on any connected component of this complement.*

We finish this chapter by describing some properties of the harmonic bundle $p_*\mathcal{C}^{\infty h}(\widehat{H})$ associated to a nilpotent orbit or Sabbah orbit (here $p : \mathbb{P}^1 \times \Delta^* \to \Delta^*$ is the projection). Any harmonic bundle (E, D, C, h) on a punctured disk with coordinate r is called *tame* iff the eigenvalues of $C_{r\partial_r} \in \mathcal{E}nd(E)$ are bounded at zero (see [Sim90]). In our situation, the endomorphism $C_{r\partial_r}$ of the harmonic bundle associated to a nilpotent orbit or Sabbah orbit $(G, \nabla, G'_{\mathbb{R}}, P, w)$ is given by $C_{r\partial_r} = [r\nabla_{z\partial_r}] \in \mathcal{E}nd_{\mathcal{O}_M}(G_{|z=0})$. Recall also that the pole part $\mathcal{U}$ is defined as $\mathcal{U} = [z\nabla_{z\partial_z}] \in \mathcal{E}nd_{\mathcal{O}_M}(G_{|z=0})$.

Lemma 4.6.

1. *The harmonic bundle associated to a nilpotent orbit of TERP-structures is tame iff the pole part $\mathcal{U}$ of the TERP-structure is nilpotent.*

2. *The harmonic bundle associated to a Sabbah orbit of TERP-structures is always tame.*

Proof. Let $(H, \nabla, H'_{\mathbb{R}}, P, w)$ be a TERP-structure inducing a nilpotent orbit $G := \pi^*H$, then the fact that $\nabla_{z\partial_z - r\partial_r}\pi^{-1}\mathcal{O}(H) = 0$ implies that $C_{r\partial_r} = \mathcal{U} \in \mathcal{E}nd_{\mathcal{O}_M}(G_{|z=0})$. If $(H, \nabla, H'_{\mathbb{R}}, P, w)$ induces a Sabbah orbit, then the same argument shows $C_{r\partial_r} = -\mathcal{U} \in \mathcal{E}nd_{\mathcal{O}_M}(G_{|z=0})$. Therefore, it suffices in both cases to study the behavior of the eigenvalues of $\mathcal{U}$.
If G is a nilpotent orbit, then for any $r \in M$ we have that $G_{|\mathbb{C}\times\{r\}} = \pi_r^*H$, which implies that $\mathcal{U}_{|r} = r^{-1}\cdot\rho_{0,r}^{-1} \circ \mathcal{U}_{|1} \circ \rho_{0,r}$ whereas for a Sabbah orbit, $G_{|\mathbb{C}\times\{r\}} = \pi_{r^{-1}}^*H$ so that $\mathcal{U}_{|r} = r\cdot\rho_{0,r}^{-1} \circ \mathcal{U}_{|1} \circ \rho_{0,r}$. This shows that in the first case, the eigenvalues are bounded iff $\mathcal{U}_{|1}$ is nilpotent whereas in the second case they tend to zero as r approaches the origin. $\square$

5 PMHS and integrable PMTS

This chapter is devoted to establish correspondences between (polarized) mixed Hodge structures and some particular TERP-structures resp. integrable twistors. This is merely an extension to integrable twistors of the correspondence due to Simpson ([Sim97] and [Moc07]). This will allow us to use Mochizuki's main result ([Moc07, theorem 12.1], the theorem of the limit mixed twistor structure) in chapter 6.

We start by giving a very brief reminder on how to associate linear algebra data to the "topological part" of a TERP-structure. The term "topological" refers to the following fact: If a TERP-structure $(H,\nabla,H'_{\mathbb{R}},P,w)$ arises from singularity theory (see chapter 11), then the restriction $H':=H_{|\mathbb{C}^*}$ is a topological invariant of the singularity and the extension H to a bundle on $\mathbb{C}$ with meromorphic connection is of transcendental nature. We thus stick to $(H',\nabla,H'_{\mathbb{R}},P,w)$ for a moment. The following objects are either well known or have been discussed extensively in [Her03, 7.2].

- $H^\infty\supset H^\infty_{\mathbb{R}}$: the spaces of complex resp. real flat multivalued global sections of (H',∇).
- $M\in Aut(H^\infty_{\mathbb{R}})$: the monodromy of ∇, which decomposes into $M=M_s\cdot M_u$ with M_s semi-simple and M_u unipotent, $N:=\log(M_u)$ the nilpotent part.
- For any $\lambda\in\mathbb{C}$, the generalized eigenspace H^∞_λ of M for the eigenvalue λ and the corresponding flat subbundle $H'_\lambda\subset H'$. Moreover, let $H^\infty_{\arg=0}=\oplus_{\arg(\lambda)=0}H^\infty_\lambda$, $H^\infty_{\arg\neq 0}=\oplus_{\arg(\lambda)\neq 0}H^\infty_\lambda$ and $H^\infty_{\neq 1}=\oplus_{\lambda\neq 1}H^\infty_\lambda$.
- Elementary holomorphic sections of H': fix $A\in H^\infty_\lambda$ and $\alpha\in\mathbb{C}$ with $\lambda=e^{-2\pi i\alpha}$, then define $es(A,\alpha):=z^{\left(\alpha Id-\frac{N}{2\pi i}\right)}A(z):=e^{(2\pi i\alpha Id-N)\zeta}A(\zeta)$ with ζ a coordinate on the universal covering of $\mathbb{C}^*$ such that $e^{2\pi i\zeta}=z$; the space of elementary sections of order α is denoted by C^α, and sending $A\mapsto es(a,\alpha)$ defines an isomorphism $\psi_\alpha: H^\infty_{e^{-2\pi i\alpha}}\to C^\alpha$. The connection acts as follows: $\nabla_{z\partial_z}es(A,\alpha)=\alpha\cdot es(A,\alpha)-es\left(\frac{N}{2\pi i}A,\alpha\right)$.
- Polarizations: The pairing P induces two bilinear forms on H^∞, which are equivalent to each other. The first one, called L here, corresponds to the Seifert form in singularity theory: first fix any $z\in\mathbb{C}^*$ and define $L:H_z\times H_z\to\mathbb{C};(a,b)\mapsto P(a,\gamma_\pi(b))$, where γ_π is the counter-clockwise flat shift from the fibre at z to the fibre at $-z$. It is readily checked that L is monodromy invariant, so that we get a pairing on H^∞. Then the following formulas, which might seem artificial at first sight, define a pairing S on $H^\infty_{\mathbb{R}}$:
$$\begin{aligned} S(a,b) &:= (-1)(2\pi i)^w L\left(a,\frac{1}{M-Id}b\right) && \forall a,\forall b\in H^\infty_{\arg\neq 0},\\ S(a,b) &:= (2\pi i)^w L\left(a,\frac{2\pi i(-\beta)Id+N}{M-Id}b\right) && \forall a\in H^\infty,\forall b\in H^\infty_{e^{-2\pi i\beta}} \end{aligned}\tag{5.1}$$
where $\beta\in i\mathbb{R}$. If $\beta=0$ (i.e., $b\in H_1$) we put
$$\frac{2\pi i(-\beta)+N}{M-Id}:=\left(\sum_{k\geq 1}\frac{1}{k!}N^{k-1}\right)^{-1}.$$
The pairing S is monodromy invariant, nondegenerate , $(-1)^{w-1}$-symmetric on $H_{\arg\neq 0}$ and $(-1)^w$-symmetric on $H_{\arg=0}$. It takes real values on $H_{\mathbb{R}}$.
- A topological version of the Fourier-Laplace transformation: Let $G^{(\alpha)}$ for α with $\Re(\alpha)>0$ be the automorphism of $H^\infty_{e^{-2\pi i\alpha}}$ defined as:
$$G^{(\alpha)}:=\sum_{k\geq 0}\frac{1}{k!}\Gamma^{(k)}(\alpha)\left(\frac{-N}{2\pi i}\right)^k=:\Gamma\left(\alpha\cdot Id-\frac{N}{2\pi i}\right).$$
Here $\Gamma^{(k)}$ is the k-th derivative of the gamma function. For notational convenience, we let
$$G:=\sum_{\Re(\alpha)\in(0,1]}G^{(\alpha)}\in Aut\left(H^\infty=\oplus_\alpha H^\infty_{e^{-2\pi i\alpha}}\right).\tag{5.2}$$
The following identities hold true.

– Let τ be another coordinate on $\mathbb{C}$. Then

$$\int_0^{\infty\cdot z} e^{-\tau/z}\cdot es(A,\alpha-1)(\tau)d\tau = es(G^{(\alpha)}A,\alpha)(z). \tag{5.3}$$

Here the left hand side means that the values of the section for different τ's are shifted using ∇ to the fibre over z and then summed up (as an integral).

– Let $\alpha,\beta\in(0,1)+i\mathbb{R}$ and $A\in H^\infty_{e^{-2\pi i\alpha}}, B\in H^\infty_{e^{-2\pi i\beta}}$. Then

$$P\left(es(G(A),\alpha),\, es(G(B),\beta)\right)(z) = z\frac{1}{(2\pi i)^{w-1}}\cdot S(A,B). \tag{5.4}$$

– Let $\alpha,\beta\in 1+i\mathbb{R}$ and $A\in H^\infty_{e^{-2\pi i\alpha}}, B\in H^\infty_{e^{-2\pi i\beta}}$. Then

$$P\left(es(G(A),\alpha),\, es(G(B),\beta)\right)(z) = z^2\frac{-1}{(2\pi i)^{w}}\cdot S(A,B). \tag{5.5}$$

In fact, the automorphism G and the pairing S were defined in [Her03, 7.2] in a slightly more restricted situation, namely, it was assumed in that paper that the eigenvalues of the monodromy are in S^1. However, the generalization we consider here can be shown by the same type of calculations.

It is also possible to go the other way round, i.e., to construct a tuple $(H',\nabla,H'_\mathbb{R},P,w)$ starting from the vector spaces $H^\infty\supset H^\infty_\mathbb{R}$, an automorphism M and a pairing S as above. The first part is obvious as a flat bundle $(H'_\mathbb{R},\nabla)$ is equivalent to the data $(H^\infty_\mathbb{R},M)$ (similarly for the complexifications). Thus the only thing to show is how to define P starting from $(H^\infty,H^\infty_\mathbb{R},M\in Aut(H_\mathbb{R}),S)$. First remark that the formulas (5.1) can be reversed to define L starting from S. Note that S takes real values on $H^\infty_\mathbb{R}$ but L sends $H^\infty_\mathbb{R}$ to $i^w\mathbb{R}$. Then put $P(a,b)(z):=L\left(a(z),\gamma_{-\pi}b(-z)\right)$. This gives a flat pairing as L is defined on the space of flat sections. The symmetry property for P follows from that of L (i.e, one needs to check that $L(a,b)=(-1)^wL(Mb,a)$ which is straightforward). We therefore arrive at the following basic result.

Lemma 5.1 (Correspondence of topological data)**.** *There is a one to one correspondence between tuples consisting of $(H^\infty,H^\infty_\mathbb{R},M,S,w)$ with the above properties and flat bundles $H'\supset H'_\mathbb{R}$ on $\mathbb{C}^*$ equipped with a flat $(-1)^w$-symmetric pairing $P:\mathcal{O}(H')\otimes j^*\mathcal{O}(H')\to\mathcal{O}_{\mathbb{C}^*}$ sending (pointwise) $H'_\mathbb{R}$ to $i^w\mathbb{R}$.*

We will gradually enrich this result to get eventually a correspondence between sums of two polarized mixed Hodge structures with an automorphism having eigenvalues in S^1 on the one hand and integrable polarized mixed twistor structures "generated by elementary sections" on the other hand. The first step is to construct a twistor from one additional piece of data, namely, a filtration. This is the result of the next lemma which is quite close to [Her03, lemma 7.12].

Lemma 5.2. *Let $(H^\infty,H^\infty_\mathbb{R},M)$ be as above and $F^\bullet$ an exhaustive decreasing filtration on H^∞. Suppose that it is invariant under M_s and satisfies $NF^\bullet\subset F^{\bullet-1}$. Consider the flat (complex) bundle H' on $\mathbb{C}^*$ corresponding to (H^∞,M). Define the following sheaves*

$$\begin{aligned}
\mathcal{H} &:= \sum_{p\in\mathbb{Z},\Re(\alpha)\in(0,1]} \mathcal{O}_\mathbb{C} z^{(\alpha+w-1-p)Id-\frac{N}{2\pi i}}F^pH^\infty_{e^{-2\pi i\alpha}}\\
&= \sum_{\substack{p\in\mathbb{Z},\Re(\alpha)\in(0,1]\\ A\in F^pH^\infty_{e^{-2\pi i\alpha}}}} \mathcal{O}_\mathbb{C}\, es(A,\alpha+w-1-p),\\
\widetilde{\mathcal{H}} &:= \sum_{q\in\mathbb{Z},\Re(\alpha)\in(0,1]} \mathcal{O}_{\mathbb{P}^1\setminus\{0\}} z^{(q+\alpha-[\Re(\alpha)])Id-\frac{N}{2\pi i}}\overline{F}^qH^\infty_{e^{-2\pi i\alpha}}\\
&= \sum_{\substack{q\in\mathbb{Z},\Re(\alpha)\in(0,1]\\ A\in \overline{F}^qH^\infty_{e^{-2\pi i\alpha}}}} \mathcal{O}_{\mathbb{P}^1\setminus\{0\}}\, es(A,q+\alpha-[\Re(\alpha)]).
\end{aligned} \tag{5.6}$$

Theses are locally free extensions of $\mathcal{O}(H')$ to zero and infinity defining a bundle $\widehat{H} \in VB_{\mathbb{P}^1}$. The connection extends to $\widehat{H}$ with poles of order at most two at 0 and ∞, therefore $\widehat{H}$ is integrable. Twistors of this type are called generated by elementary sections.
The connection has logarithmic poles at zero and infinity if M is semi-simple. $\widehat{H}$ is pure of weight k iff $(H_{\arg\neq 0}, H_{\mathbb{R},\arg\neq 0}, F^\bullet)$ is a Hodge structure of weight $w+k-1$ and $(H_{\arg=0}, H_{\mathbb{R},\arg=0}, F^\bullet)$ is a Hodge structure of weight $w+k$.

Proof. The inclusions $\mathcal{H} \subset i_*\mathcal{O}(H')$ and $\widetilde{\mathcal{H}} \subset \widetilde{i}_*\mathcal{O}(H')$, where $i : \mathbb{C}^* \to \mathbb{C}$ and $\widetilde{i} : \mathbb{C}^* \to \mathbb{P}^1\backslash\{0\}$, are obvious. Consequently, we obtain a bundle $\widehat{H} \in VB_{\mathbb{P}^1}$. If M is the identity, then this is precisely the Rees construction ([Sim97], [Moc07]). The connection on H' extends to $\widehat{H}$ (with poles at zero and infinity) using the Leibniz rule. Then for any $A \in F^pH_{e^{-2\pi i\alpha}}$ we have

$$(z^2\nabla_z)(e^{\log(z)\left((\alpha+w-1-p)Id-\frac{N}{2\pi i}\right)}A) = z\left((\alpha+w-1-p)\,Id - \tfrac{N}{2\pi i}\right) z^{(\alpha+w-1-p)Id-\frac{N}{2\pi i}}A$$

$$= z^{(\alpha+w-p)Id-\frac{N}{2\pi i}}\left((\alpha+w-1-p)\,Id - \tfrac{N}{2\pi i}\right)A \in z^{(\alpha+w-p)Id-\frac{N}{2\pi i}}F^{p-1}H_{e^{-2\pi i\alpha}} \subset \mathcal{H}.$$

A similar calculation shows that for $A \in \overline{F}^qH_{e^{-2\pi i\alpha}}$ we have

$$z^{-2}\nabla_{\partial_{z^{-1}}}(z^{(q+\alpha-[\Re(\alpha)])Id-\frac{N}{2\pi i}}A) \in \widetilde{\mathcal{H}}.$$

In both cases the fact that $NF^\bullet \subset F^{\bullet-1}$ is essential. We see that the connection has a pole of order at most two at zero and infinity. For $N = 0$ it follows immediately that $\mathcal{H}$ is stable under $z\nabla_{\partial_z}$ and $\widetilde{\mathcal{H}}$ is stable under $z^{-1}\nabla_{\partial_{z^{-1}}}$.
In general, for $A \in (F^p \cap \overline{F}^{(w+k)-1-p})H^\infty_{arg\neq 0}$ we have that

$$z^{(\alpha+w-1-p)Id-\frac{N}{2\pi i}}A \in z^{-k}z^{((w+k)-1-p+\alpha-[\Re(\alpha)])Id-\frac{N}{2\pi i}}\overline{F}^{(w+k)-1-p}H^\infty_{e^{-2\pi i\alpha}} \subset z^{-k}\widetilde{\mathcal{H}}$$

and similarly for $A \in (F^p \cap \overline{F}^{(w+k)-p})H^\infty_{arg=0}$ (i.e., $\alpha \in 1 + i\mathbb{R}$)

$$z^{(\alpha+w-1-p)Id-\frac{N}{2\pi i}}A \in z^{-k}z^{((w+k)-p+\alpha-[\Re(\alpha)])Id-\frac{N}{2\pi i}}\overline{F}^{(w+k)-p}H^\infty_{e^{-2\pi i\alpha}} \subset z^{-k}\widetilde{\mathcal{H}}.$$

Suppose that $(H^\infty, H^\infty_{\mathbb{R}}, F^\bullet)$ is a sum of two Hodge structures of weights $w+k-1$ and $w+k$, i.e.,

$$H^\infty = \bigoplus_p \left(F^p \cap \overline{F}^{(w+k)-1-p}\right) H^\infty_{\arg\neq 0} \oplus \bigoplus_p \left(F^p \cap \overline{F}^{(w+k)-p}\right) H^\infty_{\arg=0}.$$

Then by choosing an appropriate basis, we see from the last two formulas that the constructed bundle $\widehat{H}$ is semi-stable of slope k, thus we obtain an integrable pure twistor of weight k. Conversely, if we know that $\widehat{H}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}^{rank(H)}(k)$ then reading the last calculation backwards gives us a basis of H^∞ showing that the filtration $\overline{F}^\bullet$, shifted appropriately, splits $F^\bullet$. □

Consider now the general situation of a tuple $(H^\infty, H^\infty_{\mathbb{R}}, M \in Aut(H^\infty_{\mathbb{R}}), F^\bullet H^\infty, w)$, where $F^\bullet$ and $\overline{F}^\bullet$ are not necessarily $w+k$ resp. $w+k-1$-opposed. Then the following construction is a refinement of the above correspondence.
Let N be the nilpotent part of M, and let $W_\bullet$ be the weight filtration defined by N as described in lemma 2.2, but centered at zero. By definition of $W_\bullet$, for any k the endomorphism M restricts to an element in $Aut(W_k)$.

Lemma 5.3. *The above construction applied to the tuple $(W_k, W_k \cap H^\infty_{\mathbb{R}}, M_{|W_k}, F^\bullet \cap W_k, w)$ defines a subbundle $\widehat{W}_k \subset \widehat{H}$. We obtain a filtration of $\widehat{H}$ by subbundles. There is a naturally defined nilpotent morphism $\widehat{N} : \mathcal{O}(\widehat{H}) \to \mathcal{O}(\widehat{H}) \otimes \mathcal{O}_{\mathbb{P}^1}(2)$ sending $\widehat{W}_k$ to $\widehat{W}_{k-2} \otimes \mathcal{O}_{\mathbb{P}^1}(2)$. The quotient bundle $Gr_k^{\widehat{W}}\widehat{H}$ is an integrable twistor and is canonically isomorphic to the integrable twistor corresponding to $(Gr_k^W H^\infty, Gr_k^W H^\infty_{\mathbb{R}}, M \in Aut(Gr_k^W H^\infty), F^\bullet Gr_k^W H^\infty, w)$. The induced M on $Gr_k^W H^\infty$ is semi-simple, so that $(Gr_k^{\widehat{W}}(\widehat{H}), \nabla)$ is logarithmic.*

Proof. The only point to understand is the definition of $\widehat{N}$: this is merely the functoriality of the above construction (which we will not discuss in detail here). The map $N : H^\infty \to H^\infty$ satisfies by assumption $N(F^\bullet) \subset F^{\bullet-1}$ and $N(\overline{F}^\bullet) \subset \overline{F}^{\bullet-1}$. Define $\widehat{N}es(A,\alpha) := es(NA,\alpha)$, then $\widehat{N}\mathcal{H} \subset z^{-1}\mathcal{H}$ and $\widehat{N}\widetilde{\mathcal{H}} \subset z\widetilde{\mathcal{H}}$ and, so that $\widehat{N} : \mathcal{O}(\widehat{H}) \to \mathcal{O}(\widehat{H}) \otimes \mathcal{O}_{\mathbb{P}^1}(2)$. By definition of the weight filtration $W_\bullet$ on H, $NW_k \subset W_{k-2}$, this implies the same property for $\widehat{N}$ with respect to $\widehat{W}_\bullet$. Note that the filtration $\widehat{W}$ may also be defined from $\widehat{N}$ as in lemma 2.2. $\square$

The following is now an immediate consequence of lemma 5.2.

Lemma 5.4. *$(\widehat{H}, \widehat{W}_\bullet)$ is a mixed twistor, that is, each $Gr_k^{\widehat{W}}(\widehat{H})$ is pure of weight k, if and only if the tuples $(H^\infty_{\arg\neq 0}, H^\infty_{\mathbb{R},\arg\neq 0}, W_{\bullet+w-1}, F^\bullet)$ and $(H^\infty_{\arg=0}, H^\infty_{\mathbb{R},\arg=0}, W_{\bullet+w}, F^\bullet)$ are mixed Hodge structures. Moreover, given a twistor generated by elementary sections, one recovers the filtration $F^\bullet$ on H^∞ by*

$$F^p H^\infty_{e^{-2\pi i\alpha}} := \psi_\alpha^{-1}\left(z^{p+1-w}(C^{\alpha+w-1-p} \cap \mathcal{H})\right) \tag{5.7}$$

for $\Re(\alpha) \in (0,1]$. We obtain a one to one correspondence between mixed twistors generated by elementary sections and sums of two mixed Hodge structures (on $H^\infty_{\arg\neq 0}$ and $H^\infty_{\arg=0}$).

The next step is to take into account the pairing S. We suppose a tuple $(H^\infty, H^\infty_{\mathbb{R}}, M, F^\bullet H^\infty, S)$ be given. Remember that the data $(H^\infty, H^\infty_{\mathbb{R}}, M, S)$ were already shown to be equivalent to a flat bundle (H', ∇) over $\mathbb{C}^*$ and a flat pairing P on opposite fibres. For a moment, we will only consider the extension $\mathcal{H} \in VB_{\mathbb{C}}$ (i.e., we will not make use of $\overline{F}^\bullet$) and show that if $F^\bullet$ satisfies a particular orthogonality property with respect to S, then the pairing P will have a zero of order w on the extension $\mathcal{H}$ making $(H, \nabla, H'_{\mathbb{R}}, P, w)$ into a TERP-structure. Define the filtration $\widetilde{F}^\bullet$ by $\widetilde{F}^\bullet := G^{-1}F^\bullet$ where G is the automorphism defined by formula (5.2). The said orthogonality property will be formulated using this twisted filtration $\widetilde{F}^\bullet$.

Lemma 5.5. *Let $(H^\infty, H^\infty_{\mathbb{R}}, M, F^\bullet, w, S)$ with all the properties of lemma 5.1 and 5.2 be given. Suppose that*

$$\left(\widetilde{F}^p H^\infty_{\arg\neq 0}\right)^\perp = \widetilde{F}^{w-p} H^\infty_{\arg\neq 0} \quad \textit{and} \quad \left(\widetilde{F}^p H^\infty_{\arg=0}\right)^\perp = \widetilde{F}^{w+1-p} H^\infty_{\arg=0}. \tag{5.8}$$

Here $^\perp$ is the orthogonal complement with respect to S. Then $(\mathcal{H}, \nabla, H'_{\mathbb{R}}, P, w)$ is a TERP-structure.

Proof. In view of what has been said before, we need to show that P sends the germ $\mathcal{H}_0$ to $z^w\mathcal{O}_{\mathbb{C},0}$ and is nondegenerate when multiplied by z^{-w}. So take two generators of this germ, i.e., let $A \in F^p H^\infty_{e^{-2\pi i\alpha}}, B \in F^q H^\infty_{e^{-2\pi i\beta}}$ (suppose first that $\alpha, \beta \in (0,1) + i\mathbb{R}$), and compute

$$P(es(A, \alpha+w-1-p), es(B, \beta+w-1-q))(z) = P(z^{(\alpha+w-1-p)Id-\frac{N}{2\pi i}}A, (-z)^{(\beta+w-1-q)Id-\frac{N}{2\pi i}}B)$$

$$= z^{2w-2-(p+q)}(-1)^{w-1-q} P(z^{\alpha Id-\frac{N}{2\pi i}}A, (-z)^{\beta Id-\frac{N}{2\pi i}}B)$$

$$= z^{2w-2-(p+q)}(-1)^{w-1-q}\frac{z}{2\pi i} S\left(G^{-1}A, G^{-1}B\right) = z^{2w-1-(p+q)}\frac{(-1)^{w-1-q}}{2\pi i} S\left(G^{-1}A, G^{-1}B\right).$$

As $S\left(G^{-1}A, G^{-1}B\right) \neq 0$ only if $p+q \leq w-1$, we get $P(es(A, \alpha+w-1-p), es(B, \beta+w-1-q))(z) \in z^w\mathcal{O}_{\mathbb{C},0}$. On the other hand, for any $A \in F^p H_{\arg\neq 0}$ there is a $B \in F^{w-p-1}H_{\arg\neq 0}$ with $S(G^{-1}A, G^{-1}B) \neq 0$, so that P is nondegenerate. Sections coming from elements in $H_{\arg=0}$ are treated similarly. $\square$

We will call TERP-structures as above *generated by elementary sections*. The following lemma shows that this notion is consistent with the corresponding one for twistor structures.

Lemma 5.6. *Let $(H, H'_{\mathbb{R}}, \nabla, P, w)$ be a TERP structure generated by elementary sections. Then the bundle $\widehat{H}$ constructed by the procedure from definition 3.2 is a twistor generated by elementary sections as defined in lemma 5.2.*

Proof. This is an immediate consequence of the following formula, which is shown in [Her03, (7.86)].

$$\tau\left(es(A,\alpha)\right) = es\left(\overline{A}, w - \overline{\alpha}\right). \tag{5.9}$$

$\square$

It is an easy exercise to check that the converse of lemma 5.5 holds, so that we get the following extension of the correspondence of topological data. Note that any TERP-structure generated by elementary sections canonically defines a filtration $F^\bullet$ on H^∞ by formula (5.7).

Lemma 5.7. *There is a one to one correspondence between TERP-structures generated by elementary sections and tuples $(H^\infty, H^\infty_\mathbb{R}, M, F^\bullet, S, w)$ as above such that the twisted filtration $\widetilde{F}^\bullet$ satisfies condition (5.8).*

The last step is now a (common) extension of the correspondence of lemma 5.4 and of the last result putting together (mixed) twistors and polarizations. We cite the following definition from [Moc07, definition 3.48]. We restrict here to weight zero, which is what we need, but the definition extends to any weight.

Definition 5.8. *Let $(\widehat{H}, \widehat{N}, \widehat{W}_\bullet)$ be a mixed twistor structure (with $\widehat{W}_\bullet$ generated by $\widehat{N}$), suppose $\deg(\widehat{H}) = 0$, and let $\widehat{S} : \mathcal{O}(\widehat{H}) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \sigma^*\mathcal{O}(\widehat{H}) \to \mathcal{O}_{\mathbb{P}^1}$ be a non-degenerate pairing. Then $\widehat{S}$ is called a polarization if the following holds.*

- *$\widehat{S}$ is a morphism of mixed twistors and satisfies $\widehat{S}(\widehat{N}-,-) + S(-, \sigma^*\widehat{N}(-)) = 0$. This implies in particular that $\widehat{S}$ induces a morphism*

$$\begin{array}{rcl} \widehat{S}_k : \left(Gr_k^{\widehat{W}} \otimes \mathcal{O}_{\mathbb{P}^1}(-2k)\right) \otimes \left(\sigma^* Gr_k^{\widehat{W}}\right) & \longrightarrow & \mathcal{O}_{\mathbb{P}^1} \\ (a,b) & \longmapsto & \widehat{S}\left(\widehat{N}^k a, b\right). \end{array}$$

  and then also a morphism

$$\begin{array}{rcl} i^{-k}\widehat{S}_k : \left(Gr_k^{\widehat{W}} \otimes \mathcal{O}_{\mathbb{P}^1}(-k)\right) \otimes \sigma^* \left(Gr_k^{\widehat{W}} \otimes \mathcal{O}_{\mathbb{P}^1}(-k)\right) & \longrightarrow & \mathcal{O}_{\mathbb{P}^1} \\ (a,b) & \longmapsto & i^{-k}\widehat{S}_k(a,b). \end{array}$$

  The term i^{-k} comes from the identification $\sigma^*\mathcal{O}_{\mathbb{P}^1}(-k) \cong \mathcal{O}_{\mathbb{P}^1}(-k)$.

- *Let $\widehat{P}_k := \mathcal{K}er\left(\widehat{N}^{k+1} : Gr_k^{\widehat{W}} \to Gr_{-k-1}^{\widehat{W}}\right)$ be the primitive part of weight k. Then $i^{-k}\widehat{S}_k$ polarizes $\widehat{P}_k$, i.e., for all global sections $a \in H^0(\mathbb{P}^1, \widehat{P}_k \otimes \mathcal{O}_{\mathbb{P}^1}(-k))\backslash\{0\}$, $i^{-k}\widehat{S}_k(a,a) > 0$.*

$(\widehat{H}, \widehat{N}, \widehat{W}_\bullet, \widehat{S})$ is called a polarized mixed twistor structure of weight zero (PMTS).

The final result of this chapter is the following.

Lemma 5.9 (Correspondence between PMHS and integrable PMTS)**.** *There is a one to one correspondence between tuples $(H^\infty, H^\infty_\mathbb{R}, M, F^\bullet, S, w)$ with the property that $(H^\infty_{\arg\neq 0}, H^\infty_{\mathbb{R},\arg\neq 0}, -N, \widetilde{F}^\bullet, S)$ resp. $(H^\infty_{\arg=0}, H^\infty_{\mathbb{R},\arg=0}, -N, \widetilde{F}^\bullet, S)$ are PMHS's of weight $w-1$ resp. w and integrable polarized mixed twistors $(\widehat{H}, H'_\mathbb{R}, \widehat{W}_\bullet, \widehat{N}, \widehat{S}, \nabla)$ with flat real substructure on $\mathbb{C}^*$ (and such that $\widehat{W}_\bullet$ and $\widehat{N}$ are induced by the monodromy ∇) which are generated by elementary sections. The eigenvalues of M resp. of the monodromy have absolute value one.*

Proof. Let us first show the last statement: Suppose that $(H^\infty, H^\infty_\mathbb{R}, -N, \widetilde{F}^\bullet, S)$ is a sum of two PMHS's. The semi-simple part of the monodromy acts on the primitive part of each $Gr_k^W H^\infty$ and respects the hermitian form induced by S on these spaces. This forces its eigenvalues to be in S^1. Note that the space $Gr_k^W H^\infty$ is decomposed according to formula (2.5) which implies that each eigenspace has a primitive part. On the other hand, for an integrable polarized mixed twistor $(\widehat{H}, H'_\mathbb{R}, \widehat{W}_\bullet, \widehat{N}, \widehat{S}, \nabla)$ as above it is easy to see that the semi-simple part of the monodromy gives rise to a bundle map $\widehat{M}_s \in Aut(\widehat{H})$ compatible with the weight filtration $\widehat{W}_\bullet$ and with the pairing $\widehat{S}$. The induced map on $H^0(\mathbb{P}^1, \widehat{P}_k \otimes \mathcal{O}_{\mathbb{P}^1}(-k))$ therefore respects the positive definite hermitian form $i^{-k}\widehat{S}_k$. As before, we can conclude that its eigenvalues are of absolute value one. For the remaining part of this proof, we can therefore assume that the various logarithms of the monodromy eigenvalues are real numbers.
Remark that in order to define MHS on H^∞ one can work with $\widetilde{F}^\bullet$ as well as with $F^\bullet$, because these two filtrations coincide by definition on the quotients $Gr_\bullet^W$. Therefore, we know by lemma 5.4 that $\widehat{H}$ is a mixed twistor iff the tuples $(H^\infty_{\neq 1}, (H^\infty_\mathbb{R})_{\neq 1}, \widetilde{F}^\bullet, W_{\bullet+w-1})$ resp. $(H^\infty_1, (H^\infty_\mathbb{R})_1, N, \widetilde{F}^\bullet, W_{\bullet+w})$ are mixed Hodge structures.

The orthogonality of $\widetilde{F}^\bullet$ with respect to S is equivalent to the fact that $\widehat{S}$ takes values in $\mathcal{O}_{\mathbb{P}^1}$ (this follows from lemma 5.7 and lemma 3.9), so that the correspondence is true iff the positive definiteness properties of the two polarizations are equivalent. On the one hand, we have PMHS's of weights $w-1$ resp. w iff

$$i^{p-(w-1+[\alpha]+k-p)}S(A,(-N)^k\overline{A})>0$$

for all non-vanishing classes

$$[A]\in\left(\widetilde{F}^p\cap\overline{\widetilde{F}^{w-1+[\alpha]+k-p}}\right)\left(Gr_k^W H^\infty_{e^{-2\pi i\alpha}}\right)_{prim}=\left(F^p\cap\overline{F^{w-1+[\alpha]+k-p}}\right)\left(Gr_k^W H^\infty_{e^{-2\pi i\alpha}}\right)_{prim}$$

i.e., all $[A]\neq 0$ in that space with $N^{k+1}[A]=0$.
On the other hand, $(\widehat{H},\widehat{W}_\bullet,\widehat{N},\widehat{S})$ is a PMTS iff $i^{-k}\widehat{S}_k$ is positive on classes $[es(A,\alpha+w-1-p)]\in\widehat{P}_k$, where again $A\in W_kH^\infty_{e^{-2\pi i\alpha}}$ with $[A]\in\left(F^p\cap\overline{F^{w-1+[\alpha]+k-p}}\right)\left(Gr_k^W H^\infty_{e^{-2\pi i\alpha}}\right)_{prim}\backslash\{0\}$. Therefore we need to show that for all these elements A, $i^{p-(w-1+[\alpha]+k-p)}S(A,(-N)^k\overline{A})>0$ is equivalent to $i^{-k}\widehat{S}(\widehat{N}^kes(A,\alpha+w-1-p),es(A,\alpha+w-1-p))>0$. This is a consequence of the following computation.

$$\begin{aligned}
&i^{-k}\widehat{S}\left(\widehat{N}^kes(A,\alpha+w-1-p),es(A,\alpha+w-1-p)\right)\\
&=i^{-k}z^{-w}P\left(es(N^kA,\alpha+w-1-p),es(\overline{A},p+1-\alpha)\right)\\
&=i^{-k}(-1)^{p+1}(-z)^{-1-[\alpha]}P\left(es(N^kA,\alpha),es(\overline{A},1+[\alpha]-\alpha)\right)=\tfrac{i^{-k}(-1)^p}{(2\pi i)^{w-1+[\alpha]}}S\left(G^{-1}N^kA,G^{-1}\overline{A}\right)\\
&=\tfrac{i^{p-(w+k-1+[\alpha]-p)}}{(2\pi)^{w-1+[\alpha]}}S\left(G^{-1}A,(-N)^kG^{-1}\overline{A}\right)=\tfrac{1}{\Gamma(1+[\alpha]-\alpha)\Gamma(\alpha)}\cdot\tfrac{i^{p-(w+k-1+[\alpha]-p)}}{(2\pi)^{w-1+[\alpha]}}\cdot S\left(A,(-N)^k\overline{A}\right).
\end{aligned}$$

The last equality is a result of the following three facts, which are deduced directly from the definitions.

$$G^{-1}A\equiv\frac{1}{\Gamma(\alpha)}A\mod W_{k-1}\quad;\quad G^{-1}\overline{A}\equiv\frac{1}{\Gamma(1+[\alpha]-\alpha)}A\mod W_{k-1}\quad;\quad S(W_k,W_{-k-1})=0$$

This shows the equivalence of the positive definiteness properties of the two polarizations. □

6 Regular singular TERP-structures

In this chapter we investigate more closely the case of TERP-structures $(H,H'_\mathbb{R},\nabla,P,w)$ and nilpotent orbits of them which are *regular singular*, i.e., such that ∇ has a regular singularity on $\mathcal{O}(H)$ at zero (see the definition below). In this case we will obtain a direct generalization of Schmid's correspondence (theorem 2.5). This generalization will consist in a correspondence between two types of TERP-structures: those inducing nilpotent orbits and those which define polarized mixed Hodge structures on H^∞ (theorem 6.6). To start with, we recall the definition of some well-known objects associated to regular singular connections, namely, the Kashiwara-Malgrange filtration and the Deligne lattices V^α. More precisely, let (H',∇) be a flat bundle over $\mathbb{C}^*$. Remember the definition of the subspaces $C^\alpha\subset\mathcal{O}(H')$ from chapter 5. Recall also that we have chosen a total order on $\mathbb{C}$ which will be used below.

Definition 6.1. *Put*

$$V^\alpha:=\sum_{\beta\geq\alpha}\mathcal{O}_\mathbb{C}\cdot C^\beta=\bigoplus_{\alpha\leq\beta<\alpha+1}\mathcal{O}_\mathbb{C}\cdot C^\beta$$

$$V^{>\alpha}:=\sum_{\beta>\alpha}\mathcal{O}_\mathbb{C}\cdot C^\beta=\bigoplus_{\alpha<\beta\leq\alpha+1}\mathcal{O}_\mathbb{C}\cdot C^\beta$$

$$V^{>-\infty}:=\sum_{\beta}\mathcal{O}_\mathbb{C}\cdot C^\beta$$

V^α and $V^{>\alpha}$ are locally free $\mathcal{O}_\mathbb{C}$-modules whereas $V^{>-\infty}$ is locally free over $\mathcal{O}_\mathbb{C}[z^{-1}]$, i.e., a meromorphic bundle. Obviously, all of them are subsheaves of $i_*\mathcal{O}(H')$. The decreasing filtration defined by V^α on $V^{>-\infty}$ is the Kashiwara-Malgrange or V-filtration. There is a natural identification $Gr_V^\alpha:=V^\alpha/V^{>\alpha}\cong C^\alpha$.

The following definition introduces the notions of regular singular connections and explains some important object and invariants attached to them. The idea is essentially due to Varchenko ([Var80], [SS85]) who used it to define a filtration making up a mixed Hodge structure on the cohomology of the Milnor fibre of an isolated hypersurface singularity (see corollary 6.4).

Definition 6.2.
- *Let H be a bundle over $\mathbb{C}$ and ∇ a meromorphic connection with pole at zero. Then ∇ is said to have a regular singularity at zero if $\mathcal{H}$ is a subsheaf of $V^{>-\infty}$, in other words, if the sections in $\mathcal{H}$ have moderate growth at the origin when expressed in a basis of flat sections. A TERP-structure $(H, \nabla, H'_{\mathbb{R}}, P, w)$ is called regular singular if (H, ∇) is so. For a regular singular connection, the V-filtration induces a filtration on $\mathcal{H}$ which we denote by the same letter.*
- *Any section $\omega \in \mathcal{H}_0$ can be written as a (possibly infinite) sum*
$$s = \sum_{\beta \geq \alpha} s(\omega, \beta)$$
where $s(\omega, \beta) \in C^\beta$ and $s(\omega, \alpha) \neq 0$. α is called the order of ω and the section $s(\omega, \alpha)$ the principal part of ω.
- *Let $\omega_1, \ldots, \omega_n$ be a set of generating sections for $\mathcal{H}_0$ with linearly independent principal parts $s(\omega_i, \alpha_i)$. Put*
$$\mathcal{H}^{el} := \oplus_i \mathcal{O}_{\mathbb{C}} \cdot s(\omega_i, \alpha_i) \cong \sum_\alpha \mathcal{O}_{\mathbb{C}} \cdot Gr_V^\alpha \mathcal{H}.$$
Then $\mathcal{H}^{el} \subset V^{>-\infty}$ defines a vector bundle (H^{el}, ∇) over $\mathbb{C}$ generated by elementary sections.
- *Define the spectrum $Sp(H, \nabla)$ as a "subset of $\mathbb{C}$ with multiplicities", i.e.:*
$$Sp(H, \nabla) = \sum_{\alpha \in \mathbb{C}} \nu(\alpha) \alpha \in \mathbb{Z}[\mathbb{C}] \quad \text{with} \quad \nu(\alpha) := \dim_{\mathbb{C}} \left(\frac{Gr_V^\alpha \mathcal{H}}{Gr_V^\alpha z \mathcal{H}} \right)$$
We order the (possibly repeated) spectral numbers $\alpha_1, \ldots, \alpha_{rank(H)}$ by the total order on $\mathbb{C}$ defined in the beginning. (In [Her03, 7.2] the spectral numbers are called exponents.)

Lemma 6.3. *Let $(H, H'_{\mathbb{R}}, \nabla, P, w)$ be a regular singular TERP-structure. Then $(H^{el}, H'_{\mathbb{R}}, \nabla, P, w)$ is also a TERP-structure, generated by elementary sections, therefore also regular singular. We denote by $(\widehat{H^{el}}, \widehat{W}_\bullet, \widehat{N}, \widehat{S})$ the corresponding twistor. The spectral numbers obey the symmetry $\alpha_i + \alpha_{rank(H)+1-i} = w$.*

Proof. The first thing to show is that the connection has still a pole of order at most two on $\mathcal{H}^{el}$: This follows directly from the definition of the V-filtration. Namely, let $\omega \in V^\alpha \cap \mathcal{H}$ be a section having principal part $s(\omega, \alpha)$. If $z\nabla_{z\partial_z} s(\omega, \alpha) \in C^{\alpha+1}$ is non-zero, then it must be the principal part of $z\nabla_{z\partial_z}\omega \in V^{\alpha+1} \cap \mathcal{H}$. This proves that $z\nabla_{z\partial_z} s(\omega, \alpha) \in \mathcal{H}^{el}$ as required. The main point now is to see that the pairing P sends $\mathcal{H}^{el}$ to $z^w \mathcal{O}_{\mathbb{C}}$ and that $z^{-w}P$ is nondegenerate on $\mathcal{H}^{el}$. Then we know that $(H^{el}, H'_{\mathbb{R}}, \nabla, P, w)$ is a TERP-structure. By definition it is generated by elementary sections. This implies its regularity as elementary sections have moderate growth. The symmetry property of the spectrum will come out of the proof as a by-product.
First note that because $z^{-w}P$ is nondegenerate on $\mathcal{H}$, we have an isomorphism $(z^{-w}\mathcal{H}, \nabla) \cong j^*(\mathcal{H}^*, \nabla^*)$, where $\mathcal{H}^* := \mathcal{H}om_{\mathcal{O}_{\mathbb{C}}}(\mathcal{H}, \mathcal{O}_{\mathbb{C}})$ is the dual module with the induced connection and $j : z \mapsto -z$ as before. This implies that $Gr_V^\alpha(\mathcal{H}/z\mathcal{H}) \cong Gr_{V^*}^{\alpha-w}(\mathcal{H}^*/z\mathcal{H}^*)$. On the other hand, for any bundle (H, ∇) with regular singular connection, we have that $Gr_{V^*}^\alpha(\mathcal{H}^*/z\mathcal{H}^*) \cong \left(Gr_V^{-\alpha}(\mathcal{H}/z\mathcal{H})\right)^*$. This can be shown as in [Sab, Remark 3.6] or [Sab02, III.1.18] (Note that the indices must be shifted by one compared to loc.cit.). Putting these two equations together, we conclude that P induces a non-degenerate pairing
$$P : Gr_V^\alpha \frac{\mathcal{H}}{z\mathcal{H}} \otimes Gr_V^{w-\alpha} \frac{\mathcal{H}}{z\mathcal{H}} \longrightarrow z^w \mathbb{C}, \tag{6.1}$$
which gives, by the identification
$$\sum_a \mathcal{O}_{\mathbb{C}} Gr_V^\alpha \frac{\mathcal{H}}{z\mathcal{H}} \cong \frac{\sum_\alpha \mathcal{O}_{\mathbb{C}} Gr_V^\alpha \mathcal{H}}{\sum_\alpha z\mathcal{O}_{\mathbb{C}} Gr_V^\alpha \mathcal{H}} \cong \frac{\mathcal{H}^{el}}{z\mathcal{H}^{el}}$$

the desired non degenerateness of $z^{-w}P$ on $\mathcal{O}(H^{el})$. In particular, (6.1) implies that $\dim_{\mathbb{C}}\left(Gr_V^{\alpha}\frac{\mathcal{H}}{z\mathcal{H}}\right) = \dim_{\mathbb{C}}\left(Gr_V^{w-\alpha}\frac{\mathcal{H}}{z\mathcal{H}}\right)$, which is precisely the symmetry property of the spectrum. $\square$

The next result is a rather trivial but important consequence of the last lemma.

Corollary 6.4. *Let $(H, H'_{\mathbb{R}}, P, w)$ be a TERP-structure. Then the formula*

$$F^p H^{\infty}_{e^{-2\pi i\alpha}} := \psi_\alpha^{-1}\left(z^{p+1-w}Gr_V^{\alpha+w-1-p}\mathcal{H}\right) \qquad \text{with} \quad \Re(\alpha)\in(0,1] \tag{6.2}$$

defines a filtration $F^\bullet$ on the space H^∞ coinciding with the one obtained from $(H^{el}, H'_{\mathbb{R}}, \nabla, P, w)$ using lemma 5.7, i.e., $F^pH^{\infty}_{e^{-2\pi i\alpha}} = \psi_\alpha^{-1}\left(z^{p+1-w}(C^{\alpha+w-1-p}\cap\mathcal{H}^{el})\right)$. In particular, the orthogonality conditions (5.8) for the twisted filtration $\widetilde{F}^\bullet$ are satisfied, so that $\widetilde{F}^\bullet$ gives an element in $\check{D}$, the classifying space of Hodge-like filtrations on H^∞ from chapter 1.

Remark: In case a TERP-structure arises as Fourier-Laplace transform of the Brieskorn lattice of an isolated hypersurface singularity (see chapter 11), $\widetilde{F}^\bullet$ is the filtration defined by Steenbrink [SS85] as part of a polarized mixed Hodge structure on the cohomology of the Milnor fibre (the space H^∞).

The following lemma motivates the introduction of nilpotent orbits of TERP-structures. Let $(G, G'_{\mathbb{R}}, \nabla, P, w)$ be a variation of TERP-structures on Δ^* such that $\mathcal{O}(G)$ is stable under $\nabla_{z\partial_z - r\partial_r}$. First it follows that if the restriction $G_{\mathbb{C}\times\{r\}}$ to any $r\in\Delta^*$ is regular singular, then the whole bundle is regular singular along $z=0$. In this situation, consider the space G^∞ of flat multivalued sections of $G' := G_{|\mathbb{C}^*\times\Delta^*}$. Then $\widetilde{F}^\bullet$ induce a family of filtrations on G^∞ parameterized by the universal cover $\mathbb{H}$ of Δ^*.

Lemma 6.5. *This family satisfies $\widetilde{F}^\bullet(\rho) = \exp(-\rho N)\widetilde{F}^\bullet(0)$ (with ρ a coordinate on $\mathbb{H}$ such that $r = e^{2\pi i\rho}$). In particular, it yields a holomorphic map $\phi : \mathbb{H}\to\check{D}$ into the classifying space, and this map is a nilpotent orbit of Hodge structures with nilpotent endomorphism $-N$ iff $\widetilde{F}^\bullet(\rho)\in D$ for $\Im(\rho)\gg 0$.*

Proof. The first statement is shown in [Her03, theorem 7.20]. It implies that the dimensions of the various $\widetilde{F}^p$'s are constant in ρ and by the last corollary we know that $\widetilde{F}^\bullet(\rho)$ satisfies the orthogonality conditions for any ρ. This proves the second statement. $\square$

The following theorem is the main result of this chapter. It generalizes the correspondence 2.5.

Theorem 6.6. *Let $(H, \nabla, H'_{\mathbb{R}}, P)$ be a regular singular TERP-structure. The following two conditions are equivalent.*

1. *$(H, \nabla, H'_{\mathbb{R}}, P)$ induces a nilpotent orbit.*
2. *$(H^\infty, H^\infty_{\mathbb{R}}, -N, S, \widetilde{F}^\bullet)$ defines a PMHS of weight $w-1$ resp. w on $H^\infty_{\arg\neq 0}$ resp. on $H^\infty_{\arg=0}$.*

Before entering into the proof, let us make some comments on the content of this result. Suppose that 2.) holds, then the family $\widetilde{F}^\bullet(\rho)$ will eventually end up in the interior D of the classifying space by Schmid's correspondence, which by chapter 5 amounts to say that $(\widehat{G^{el}}, \widehat{S})$ is a family of pure polarized twistors. However, it is by no means obvious that the same holds for $(\widehat{G}, \widehat{S})$. It was shown in [Her03, theorem 7.20] that this is indeed the case, mainly because the two objects $\widehat{G^{el}}$ and $\widehat{G}$ tend "to each other" in a suitable sense when r approaches the origin. Therefore 2.) –¿ 1.) of the above theorem is precisely what has been shown in [Her03, theorem 7.20]. We need to prove the converse. The same difficulty occur: From the fact that G underlies a variation of polarized pure TERP-structure we know (almost by definition) that $(\widehat{G}, \mathbb{D}, \widehat{S})$ is a VPTS, but it is not at all clear a priori that this is also true for $(\widehat{G^{el}}, \mathbb{D}, \widehat{S})$. The proof will actually be quite different: We will use one of the main results of [Moc07] to obtain from the variation G a "limit object" which we can identify with the twistor $(\widehat{H^{el}}, \widehat{W}_\bullet, \widehat{N}, \widehat{S})$ constructed from our original TERP-structure. Mochizuki's theorem says that this limit is a PMTS so that by the correspondence of lemma 5.9 we can conclude that $(-N, \widetilde{F}^\bullet)$ gives rise to a (sum of) PMHS on H^∞.

In order to state the next theorem, recall from definition 3.11 that any VPTS $(\widehat{G}, \mathbb{D}, \widehat{S})$ of weight zero on a base space M gives rise to the structure of a harmonic bundle on $E = p_*\mathcal{C}^{\infty h}(\widehat{G})$, where $p : \mathbb{P}^1\times M\to M$ is

the projection. If M a punctured disk, Mochizuki constructs (following the idea in [Sim97, section 6]) a "limit" twistor (i.e., a vector bundle on $\mathbb{P}^1 \times \{0\}$). It should be noticed that in the one dimensional situation we are looking at, the limit object is already considered in the work of Simpson. Mochizuki actually works in a much more general situation, namely, he considers harmonic bundles in any dimensions defined on the complement of a normal crossing divisor. As we will see, the nilpotent orbit assumption also restricts the class of harmonic bundles which can occur.
To be more precise, Mochizuki defines for any $a \in (0,1]$ a tuple

$$\left(S^{can}_{(a,0)}(E), \widehat{\mathcal{N}}, \widehat{\mathcal{W}}_\bullet, \widehat{\mathcal{S}}\right)$$

where $S^{can}_{(a,0)}(E) \in VB_{\mathbb{P}^1\times\{0\}}$, $\widehat{\mathcal{N}}$ is a nilpotent bundle endomorphism generating a weight filtration $\widehat{\mathcal{W}}_\bullet$ and $\widehat{\mathcal{S}}$ is a pairing as the one discussed in chapter 5. Theorem 12.1 in [Moc07] then states that $\left(S^{can}_{(a,0)}(E), \widehat{\mathcal{N}}, \widehat{\mathcal{W}}_\bullet, \widehat{\mathcal{S}}\right)$ is a polarized mixed twistor structure. Therefore, the proof of theorem 6.6 reduces by lemma 5.9 to the following comparison result.

Theorem 6.7. *Consider the harmonic bundle $E := p_*\mathcal{C}^{\infty h}(\widehat{G})$ on Δ^*, where $G := \pi^* H$ is a nilpotent orbit of TERP-structures. Then $\widehat{H^{el}} \in VB_{\mathbb{P}^1}$ is canonically isomorphic to the bundle $\bigoplus_{0<a\leq 1} S^{can}_{(a,0)}(E)$ and this isomorphism identifies $\widehat{W}_\bullet, \widehat{S}$ from lemma 6.3 with the objects $\widehat{\mathcal{W}}_\bullet, \widehat{\mathcal{S}}$ and sends $\widehat{N}$ to $2\pi \cdot \widehat{\mathcal{N}}$.*

Proof:
We will mainly have to adapt the proof of [Moc07, theorem 12.1] to our situation. In fact, as the setup considered by Mochizuki is much more general, everything will simplify to a large extent but still the abundance of objects and notations in [Moc07] makes this translation somewhat painful. We will proceed in three steps.

(I) Canonical extensions over $r = 0$ and (z, r)-elementary sections
Let us consider for a moment the restriction $G' := G_{|\mathbb{C}^*\times\Delta^*}$ of our family of TERP-structures. We will use a slight generalization of the constructions of the beginning of chapter 5 to the case of a flat bundle on the complement of a normal crossing divisor. The divisor here is just $\{r = 0\} \cup \{z = 0\} \subset \mathbb{C} \times \Delta$, our bundle is G' and as has been shown, the two monodromies are equal. For each fixed z, we can consider the restriction $G_{|\{z\}\times\Delta^*}$ which is flat and we have a space of multivalued flat sections of this bundle which comes equipped with a monodromy operator. If z varies, they patch together to a flat bundle $\mathcal{H}^\infty$ on $\mathbb{C}^* \times \{0\}$, on which the horizontal monodromy acts. We denote by $\mathcal{H}^\infty_\lambda$ the flat generalized eigenbundle for this monodromy. It is obviously the same as the flat subbundle for the "intrinsic" vertical monodromy of $\mathcal{H}^\infty$. For any fixed r, $(\mathcal{H}^\infty, \nabla)$ can be identified with the restriction $(G'_{|\mathbb{C}^*\times\{r\}}, \nabla_z)$, in particular, $(\mathcal{H}^\infty, \nabla)$ is isomorphic to (H', ∇), the restriction over $\mathbb{C}^*$ of the original TERP-structure.
The bundle $\mathcal{H}^\infty$ can also be defined more intrinsically by putting $\mathcal{H}^\infty := \psi_r(G'^\nabla)$, where G'^∇ is the local system on $\mathbb{C}^* \times \Delta^*$ corresponding to the flat bundle G', and ψ is the functor of nearby cycles of Deligne (see [Dim04]). In this simple situation, the complex $\psi_r(G'^\nabla)$ has cohomology only in degree zero. This implies directly that $\mathcal{H}^\infty$ is a local system on $\{r = 0\} = \mathbb{C}^* \times \{0\}$ with monodromy, and by abuse of notation we denote by $(\mathcal{H}^\infty, \nabla)$ the flat bundle corresponding to it and by $\mathcal{H}^\infty_\lambda$ the eigenbundle of the monodromy operator. Note that as we have a canonical identification of local systems

$$(\overline{\gamma^* G'})^{\gamma^*\nabla} \cong (G')^\nabla$$

we can also describe $\mathcal{H}^\infty$ by $\psi_r((\overline{\gamma^* G'})^{\gamma^*\nabla})$. This fact will be used in a moment. As before in the absolute situation, we define for any $\alpha \in \mathbb{C}$ by $\mathcal{V}^\alpha$ the bundle

$$\mathcal{V}^\alpha = \bigoplus_{\alpha\leq\beta<\alpha+1} \mathcal{O}_{\mathbb{C}^*\times\Delta} r^{\beta Id - \frac{N}{2\pi i}} \mathcal{H}^\infty_{e^{-2\pi i\beta}}$$

By definition, $\mathcal{V}^\alpha$ is the unique $\mathcal{O}_{\mathbb{C}^*\times\Delta}$-free extension of $\mathcal{O}(G')$ having a logarithmic pole along $\mathbb{C}^* \times \{0\}$ with residue eigenvalues β such that $\alpha \leq \beta < \alpha + 1$. There is a natural isomorphism

$$\mathcal{H}^\infty_{e^{-2\pi i\alpha}} \longrightarrow Gr^\alpha_{\mathcal{V}} := \mathcal{V}^\alpha / \mathcal{V}^{>\alpha}$$

where, as before $\mathcal{V}^{>\alpha} := \bigcup_{\beta>\alpha} \mathcal{V}^{\beta}$. Similarly, one defines

$$\overline{\mathcal{V}}^{\alpha} := \bigoplus_{\alpha \leq \beta < \alpha+1} \mathcal{O}_{\mathbb{C}^* \times \overline{\Delta}} \overline{r}^{\overline{\beta} Id + \frac{N}{2\pi i}} \mathcal{H}^{\infty}_{e^{2\pi i \overline{\beta}}}$$

yielding an isomorphism $\mathcal{H}^{\infty}_{e^{2\pi i \overline{\alpha}}} \to Gr^{\alpha}_{\overline{\mathcal{V}}}$. In the sequel, we need an explicit description of generating sections of G, which is provided by the following lemma.

Lemma 6.8. *Consider a basis of the original TERP-structure H given by sections which we decompose into elementary sections, i.e.:*

$$\mathcal{O}(H) = \bigoplus_{i=1}^{rank(H)} \mathcal{O}_{\mathbb{C}} \left(\sum_{j \in \mathbb{N}} es(A_{ij}, \alpha_{ij}) \right)$$

with $A_{ij} \in H^{\infty}$ and $\alpha_{i1} < \alpha_{i2} < \ldots$ for all i. Then we have

$$\mathcal{O}(G) = \bigoplus_{i=1}^{rank(H)} \mathcal{O}_{\mathbb{C} \times \Delta^*} \left(\sum_{j \in \mathbb{N}} r^{\alpha_{ij} Id - \frac{N}{2\pi i}} es(A_{ij}, \alpha_{ij}) \right) \tag{6.3}$$

Proof. This follows directly from the definition of $G = \pi^* H$, namely,

$$\nabla_{z\partial_z - r\partial_r} \left(\sum_{j \geq 1} r^{\alpha_{ij} Id - \frac{N}{2\pi i}} es(A_i, \alpha_{ij}) \right) = 0$$

holds. $\square$

(II) KMSS-spectrum of nilpotent orbits

Let us recall in brief the notion of the parabolic filtration associated to a VPTS on Δ^*. It is mainly due to Simpson ([Sim90]), but we need a slightly more precise version as in [Moc07] (see also [Sab05b, Corollary 5.3.1]).

Definition-Theorem 6.9. *Let $(E, \overline{\partial}, \theta, h)$ be a tame harmonic bundle on Δ^*, take the pullback $E' := p^* E_{|\mathbb{C} \times \Delta^*}$ and define $\mathcal{E} := \mathcal{K}er \left(\overline{\partial} + z\overline{\theta} : \mathcal{C}^{\infty h}(E') \to \mathcal{C}^{\infty h}(E') \otimes \mathcal{O}_{\mathbb{C}} \mathcal{A}^{0,1}_{\Delta^*} \right) \in VB_{\mathbb{C} \times \Delta^*}$. Consider the following extensions to sheaves over $\mathbb{C} \times \Delta$ (let $i : \mathbb{C} \times \Delta^* \hookrightarrow \mathbb{C} \times \Delta$):*

$${}_a\mathcal{E} := \left\{ s \in i_* \mathcal{E} \mid |s|_{p^*h} \in O\left(|r|^{-\epsilon - a}\right) \ \ \forall \epsilon > 0 \right\},$$

$${}_*\mathcal{E} := \bigcup_{a \in \mathbb{R}} {}_a\mathcal{E} = \{ s \in i_* \mathcal{E} \mid |s|_{p^*h} \text{ has moderate growth along } \mathbb{C} \times \{0\} \}.$$

The increasing filtration of ${}_*\mathcal{E}$ by the subsheaves ${}_a\mathcal{E}$ is called parabolic filtration. We have that $r \cdot {}_a\mathcal{E} \cong {}_{a-1}\mathcal{E}$ endowing ${}_a\mathcal{E}$ with a $\mathcal{O}_{\mathbb{C} \times \Delta}$-module structure. The sheaf ${}_*\mathcal{E}$ is a locally free $\mathcal{O}_{\mathbb{C} \times \Delta}[r^{-1}]$-module and for any $z \in \mathbb{C}$, the restrictions ${}_a\mathcal{E}^z := j_z^*({}_a\mathcal{E})$ are $\mathcal{O}_{\Delta}$-locally free extension of $j_z^* \mathcal{E}$ over $(z, 0)$ (here $j_z : \{z\} \times \Delta \hookrightarrow \mathbb{C} \times \Delta$). However, ${}_a\mathcal{E}$ is not $\mathcal{O}_{\mathbb{C} \times \Delta}$-free in general. The z-connection $(z\partial + \theta)$ sends ${}_a\mathcal{E}^z$ to ${}_{a+1}\mathcal{E}^z$, in particular for any $z \in \mathbb{C}^*$, $\partial + z^{-1}\theta$ has a logarithmic pole on ${}_a\mathcal{E}^z$ Let $Gr^{\mathcal{P}}_a({}_*\mathcal{E}^z) := {}_a\mathcal{E}^z / {}_{<a}\mathcal{E}^z$ with ${}_{<a}\mathcal{E}^z = \bigcup_{b<a} {}_b\mathcal{E}^z$. The residue of $z\partial + \theta$ at $r = 0$ acts on the graded pieces. Denote by $\mathbb{E}_{\alpha} Gr^{\mathcal{P}}_a({}_*\mathcal{E}^z)$ its generalized eigenspace decomposition and by $\mathcal{N}^z_E$ its nilpotent part. Define

$$KMSS(\mathcal{E}^z) := \left\{ (a, \alpha) \in \mathbb{R} \times \mathbb{C} \,|\, \mathbb{E}_{\alpha} Gr^{\mathcal{P}}_a({}_*\mathcal{E}^z) \neq \{0\} \right\}$$

*to be the **Kashiwara-Malgrange-Sabbah-Simpson spectrum** of $\mathcal{E}^z$. The multiplicity of $(a, \alpha) \in KMSS(\mathcal{E}^z)$ is by definition $\dim_{\mathbb{C}} \mathbb{E}_{\alpha} Gr^{\mathcal{P}}_a(\mathcal{E}^z)$. In particular, $(a, \alpha) \in KMSS(\mathcal{E}^0)$ only if α is an eigenvalue of the residue endomorphism of the Higgs field θ. The KMSS-spectrum is determined by the reduced KMSS-spectrum*

$$KMSS_{red}(\mathcal{E}^z) := KMSS(\mathcal{E}^z) \cap (0, 1] \times \mathbb{C},$$

because it is invariant under the shift $(a, \alpha) \mapsto (a + 1, \alpha - z)$.

We quote the following properties of the KMSS-spectrum.

Lemma 6.10. *Let $(E, \overline{\partial}, \theta, h)$ be a tame harmonic bundle on Δ^*.*

1. *The KMSS-spectrum satisfies:*

$$(a,\alpha) \in KMSS(\mathcal{E}^0) \quad \text{iff} \quad (a + 2\Re(z\cdot\overline{\alpha}), \alpha - az - \overline{\alpha}z^2) \in KMSS(\mathcal{E}^z) \qquad \forall z \in \mathbb{C}$$

2. *The KMSS-spectrum of the harmonic bundle $(E, \partial, \overline{\theta}, h)$ on the conjugate complex manifold $\overline{\Delta^*}$ is given by*

$$KMSS(\overline{\mathcal{E}}^z) = \{-a + 2\Re(z^{-1}\cdot\alpha), \overline{\alpha} + az^{-1} - \alpha z^{-2}) \mid (a,\alpha) \in KMSS(\mathcal{E}^0)\} \qquad \forall z \in \mathbb{P}^1\backslash\{0\},$$

where $\overline{\mathcal{E}}$ is constructed from E as above, but using the pullback $p^*E_{|(\mathbb{P}^1\backslash\{0\})\times\overline{\Delta^*}}$.

3. *For any $(a,\alpha) \in KMSS(\mathcal{E}^0)$, there is a unique holomorphic bundle $\mathcal{G}_{(a,\alpha)}$ on $\mathbb{C}\times\{0\}$ whose restriction to $z\times\{0\}$ satisfies*

$$(\mathcal{G}_{(a,\alpha)})_{|z} = \mathbb{E}_{\alpha - az - \overline{\alpha}z^2}\left(Gr^{\mathcal{P}}_{a+2\Re(z\cdot\overline{\alpha})}({}_*\mathcal{E}^z)\right)$$

and a unique holomorphic bundle $\overline{\mathcal{G}}_{(-a,\overline{\alpha})}$ on $(\mathbb{P}^1\backslash\{0\})\times\{0\}$ whose restriction to $z\times\{0\}$ satisfies

$$(\overline{\mathcal{G}}_{(-a,\overline{\alpha})})_{|z} = \mathbb{E}_{\overline{\alpha}+az^{-1}-\alpha z^{-2}}\left(Gr^{\mathcal{P}}_{-a+2\Re(z^{-1}\cdot\alpha)}(\overline{{}_*\mathcal{E}}^z)\right)$$

for any $z \in \mathbb{P}^1\backslash\{0\}$.

Proof. The first statement is [Moc07, proposition 1.8 and corollary 7.71]. The second point then simply reduces to the fact that $KMSS(\overline{\mathcal{E}}^\infty) = \{(-a,\overline{\alpha}) \mid (a,\alpha) \in KMSS(\mathcal{E}^0)\}$ which is obvious. The third point is explained in detail in [Moc07, sections 1.3.6, 8.9.1 and 11.2.3]. □

We can apply this result to our more special situation. Remember that if a harmonic bundle E is given by $p_*\mathcal{C}^{\infty h}(\widehat{G})$ where G underlies a variation of TERP-structures on Δ^*, then the sheaf $\mathcal{E}$ from definition-theorem 6.9 is nothing else then $\mathcal{O}(G)$ and the connection $\partial + z^{-1}\theta$ on $\mathcal{E}$ is canonically identified with the horizontal part $\nabla_r : \mathcal{O}(G) \to \mathcal{O}(G) \otimes z^{-1}\mathcal{O}_\mathbb{C}\Omega^1_{\Delta^*}$ of ∇. In the same way, $(\overline{\mathcal{E}}, \overline{\partial} + z\overline{\theta})$ can be identified with $(\mathcal{O}(\overline{\gamma^*G}), \gamma^*\nabla_r)$.

Lemma 6.11. *Consider a harmonic bundle E constructed from a nilpotent orbit of TERP-structures as above.*

1. *The reduced KMSS-spectrum is $KMSS_{red}(\mathcal{E}^z) = \{(a, -z\cdot a) \mid a \in L \subset (0,1]\}$, where L is finite.*
2. *The eigenvalues of the monodromy of either ∇_{∂_z} or ∇_{∂_r} are equal to $e^{2\pi i a}$ with $a \in L$. In particular, they are elements of S^1.*
3. *For any $a \in L$, the restriction $({}_a\mathcal{E})_{|\mathbb{C}^*\times\Delta}$ coincides with the sheaf $\mathcal{V}^{-a}$.*
4. *${}_a\mathcal{E}$ is $\mathcal{O}_{\mathbb{C}\times\Delta}$-free for all $a \in L$.*

Proof.
1. For a nilpotent orbit, it was shown in lemma 4.6 that the Higgs field is nilpotent iff the polar part $\mathcal{U}$ of the vertical connection ∇_z of our original TERP-structure is nilpotent. By [Sab02, theorem II.4.1], this is the case for a regular singular connection, so that the eigenvalues of the Higgs field are zero. This means that $KMSS_{red}(\mathcal{E}^0) = \{(a,0) \mid a \in L \subset (0,1]\}$ so that by the preceding lemma we get that $KMSS_{red}(\mathcal{E}^z) = \{(a, -z\cdot a) \mid a \in L \subset (0,1]\}$.

2. It was already shown in lemma 4.2 that the nilpotent orbit condition implies that vertical and horizontal monodromy coincide. We know that $\nabla_{\partial_r} = \partial + z^{-1}\theta$ is logarithmic on any ${}_a\mathcal{E}^z$, so that the eigenvalues of the monodromy are of the form $e^{-2\pi i\gamma}$ where γ is an eigenvalue of the residue endomorphism of ∇_{∂_r} acting on ${}_a\mathcal{E}^z/(r\cdot{}_a\mathcal{E}^z)$. By the first point, the eigenvalues of the residue of the z-connection $z\nabla_{\partial_r} = z\partial + \theta$ on $Gr^{\mathcal{P}}_a({}_*\mathcal{E}^z)$ are of the form $-za$ with $a \in L$. Therefore the monodromy eigenvalues are precisely the numbers $e^{2\pi i a} \in S^1$.

3. It follows from 1. and the defining property of $\mathcal{V}^{-a}$ that ${}_a\mathcal{E}^z \cong \mathcal{V}^{-a}_{|\{z\}\times\Delta}$ for all $z \in \mathbb{C}^*$. This implies ${}_a\mathcal{E}_{|\mathbb{C}^*\times\Delta} \cong \mathcal{V}^{-a}$.

4. On $\mathbb{C} \times \Delta^*$, the sheaf ${}_a\mathcal{E}$ coincides by definition with $\mathcal{E}$ which is $\mathcal{O}_{\mathbb{C}\times\Delta^*}$-free. As was just shown, on $\mathbb{C}^* \times \Delta$ it is isomorphic to $\mathcal{V}^{-a}$ which is obviously free over $\mathcal{O}_{\mathbb{C}^*\times\Delta}$. Hence we only need to show that the germ ${}_a\mathcal{E}_{(0,0)}$ is a free $\mathcal{O}_{\mathbb{C}\times\Delta,(0,0)}$-module. Suppose first that $a \notin L+\mathbb{Z}$, then this follows from [Moc07, proposition 1.11]. If, however, $a \in L+\mathbb{Z}$, then by discreteness of $L+\mathbb{Z}$ in $\mathbb{R}$ we can choose an $\epsilon_0 > 0$ such that $(a, a+\epsilon_0) \cap (L+\mathbb{Z}) = \emptyset$. Then for any smaller $0 < \epsilon < \epsilon_0$, we have an equality
$$ {}_{a+\epsilon}\mathcal{E}_{|\mathbb{C}\times\Delta\backslash\{(0,0)\}} = {}_a\mathcal{E}_{|\mathbb{C}\times\Delta\backslash\{(0,0)\}}. $$
The sheaves ${}_{a+\epsilon}\mathcal{E}$ are all free over $\mathcal{O}_{\mathbb{C}\times\Delta}$ by construction, and a classical result ([Ser66]) asserts that for any locally free sheaf defined on a complement of a subvariety of codimension at least two, there is at most one locally free extension to the entire space. This implies that for any two $\epsilon, \epsilon' \in (0,\epsilon_0)$ as above, ${}_{a+\epsilon}\mathcal{E} = {}_{a+\epsilon'}\mathcal{E}$ on the whole of $\mathbb{C}\times\Delta$. But by definition, if $s \in {}_{a+\epsilon}\mathcal{E}$, then for any $\delta > 0$, $|s|_{p^*h} \in O(|r|^{-a-\epsilon-\delta})$. The last formula is true for any ϵ and δ because of ${}_{a+\epsilon}\mathcal{E} = {}_{a+\epsilon'}\mathcal{E}$, so that we already have $s \in {}_a\mathcal{E}$. This gives ${}_{a+\epsilon}\mathcal{E} = {}_a\mathcal{E}$ and thus the desired freeness property.

□

Remark: It turns out that the last lemma holds true in a more general context, namely, for any *integrable* VPTS over a curve with tame behavior at the singularities. In other words, the nilpotent orbit condition is needed only to get tameness of the corresponding harmonic bundle as in lemma 4.6. This follows from the simple observation that if we have an integrable VPTS, then the monodromy of $\partial + z^{-1}\theta$ is constant in z (isomonodromic situation), so that the eigenvalues of the residue of this operator along $r=0$ on $Gr_a^{\mathcal{P}}({}_*\mathcal{E})$ are equal to $\frac{\alpha}{z} - a - \overline{\alpha}z$ which can be constant only if $\alpha = 0$. Therefore θ is nilpotent which implies all other statements of the lemma.

(III) Identification of twistors

The main playing characters of this last part of the proof are the bundles $\mathcal{G}_{(a,\alpha)} \in VB_{\mathbb{C}}$ and $\mathcal{G}_{(-a,\overline{\alpha})} \in VB_{\mathbb{P}^1\backslash\{0\}}$ defined above. According to lemma 6.11, only pairs $(a,0)$ (resp. $\underline{\ }(-a,0)$) occur as indices in our situation, therefore we denote the corresponding bundles simply by $\mathcal{G}_a$ resp. $\overline{\mathcal{G}}_{-a}$. The following lemma gives an explicit description of theses objects.

Lemma 6.12. *Let E as above be constructed from a nilpotent orbit G of TERP-structures. Then we can consider the graded pieces $Gr_a^{\mathcal{P}}({}_*\mathcal{E})$ globally on $\mathbb{C}$ and there is an identification $\mathcal{G}_a \cong Gr_a^{\mathcal{P}}({}_*\mathcal{E})$ for any $a \in L$. Similarly, $\overline{\mathcal{G}}_{-a} \cong Gr_{-a}^{\mathcal{P}}({}_*\overline{\mathcal{E}})$. It follows that the restriction $(\mathcal{G}_a)_{|\mathbb{C}^*}$ is canonically isomorphic to $Gr_{\mathcal{V}}^{-a} \cong \mathcal{H}^\infty_{e^{2\pi i a}}$ and that $(\overline{\mathcal{G}}_{-a})_{|\mathbb{C}^*} \cong Gr_{\overline{\mathcal{V}}}^{a} \cong \mathcal{H}^\infty_{e^{2\pi i a}}$.*

Proof. The quotients $Gr_a^{\mathcal{P}}({}_*\mathcal{E})$ are bundles since ${}_a\mathcal{E}$ is locally free. For any fixed $z \in \mathbb{C}$, the third part of lemma 6.10 shows that $(\mathcal{G}_{(a,0)})_{|z} = \mathbb{E}_{-za}\left(Gr_a^{\mathcal{P}}({}_*\mathcal{E}^z)\right)$. But the fact that only pairs $(a,-za)$ occur as elements of $KMSS(\mathcal{E}^z)$ shows that $\mathbb{E}_{-za}\left(Gr_a^{\mathcal{P}}({}_*\mathcal{E}^z)\right) = Gr_a^{\mathcal{P}}({}_*\mathcal{E}^z)$. The same arguments applied to the sheaves $\overline{\mathcal{E}}^z$ yield $\overline{\mathcal{G}}_{-a} \cong Gr_{-a}^{\mathcal{P}}({}_*\overline{\mathcal{E}})$. Concerning the second statement, the identification $({}_a\mathcal{E})_{|\mathbb{C}^*\times\Delta} \cong \mathcal{V}^{-a}$ from lemma 6.11 shows $(\mathcal{G}_a)_{|\mathbb{C}^*} \cong Gr_{\mathcal{V}}^{-a}$. Similarly one checks that $({}_{-a}\overline{\mathcal{E}})_{|\mathbb{C}^*\times\Delta} \cong \overline{\mathcal{V}}^a$ yielding $(\overline{\mathcal{G}}_{-a})_{|\mathbb{C}^*} \cong Gr_{\overline{\mathcal{V}}}^{a}$. □

Corollary 6.13. *The morphism $\Phi_{(a,0)}^{can}$ from [Moc07, section 10.4.1] can be expressed as the composition of the isomorphisms*
$$ \mathcal{H}^\infty_{e^{2\pi i a}} \longrightarrow Gr_{\mathcal{V}}^{-a} \longrightarrow \left(Gr_a^{\mathcal{P}}({}_*\mathcal{E}) \cong \mathcal{G}_a\right)_{|\mathbb{C}^*} $$
Similarly, the morphism $\Phi^{\dagger\, can}_{(-a,0)}$ is given by
$$ \mathcal{H}^\infty_{e^{2\pi i a}} \longrightarrow Gr_{\overline{\mathcal{V}}}^{a} \longrightarrow \left(Gr_{-a}^{\mathcal{P}}({}_*\overline{\mathcal{E}}) \cong \overline{\mathcal{G}}_{-a}\right)_{|\mathbb{C}^*}. $$

Proof. This is a direct consequence of the last lemma as soon as we know that the filtration $\mathcal{F}_\bullet$ on $\mathcal{H}^\infty$ considered by Mochizuki is trivial in our situation which follows from the behavior of the graded pieces of this filtration (the "KMSS-spectrum of flat multivalued sections"), see [Moc07, proof of lemma 2.4, sections 7.4.2 and 9.1.5]. □

The next result is the central step in the comparison of Mochizuki's objects with the ones we are interested in. Note that by the last lemma, we can consider $\mathcal{G}_a$ as a subsheaf of $i_*\mathcal{H}^\infty_{e^{2\pi i a}}$ and $\overline{\mathcal{G}}_{-a}$ as a subsheaf of $\widetilde{i}_*\mathcal{H}^\infty_{e^{2\pi i a}}$, where, as before, $i : \mathbb{C}^* \hookrightarrow \mathbb{C}$ and $\widetilde{i} : \mathbb{C}^* \hookrightarrow \mathbb{P}^1\backslash\{0\}$.

Lemma 6.14. *Consider an identification $\mathcal{H}^\infty_{e^{2\pi ia}} \longrightarrow \mathcal{O}(H'_{e^{2\pi ia}})$. Under this isomorphism, $\mathcal{G}_a$ is mapped to*

$$\sum_{k\in\mathbb{Z}} \mathcal{O}_\mathbb{C}\left(Gr_V^{k-a}\mathcal{O}(H)\right) \subset i_*\mathcal{O}(H'_{e^{2\pi ia}})$$

Therefore, $\oplus_{a\in L}\mathcal{G}_a \cong \mathcal{O}(H^{el})$. Similarly, $\oplus_{a\in L}\overline{\mathcal{G}}_{-a} \xrightarrow{\cong} \mathcal{O}(\widehat{H}^{el}_{|\mathbb{P}^1\setminus\{0\}}) = \mathcal{O}(\overline{\gamma^* H^{el}}) \subset \widetilde{i}_*\mathcal{O}(H')$.

Proof. As was shown in part (I)

$$\mathcal{O}(G) = \bigoplus_{i=1}^{rank(H)} \mathcal{O}_{\mathbb{C}\times\Delta^*}\left(\sum_{j\geq 1} r^{\alpha_{ij}Id - \frac{N}{2\pi i}} es(A_{ij},\alpha_{ij})\right)$$

Lemma 6.11 gives

$$_a\mathcal{E} = \bigoplus_{i=1}^{rank(H)} \mathcal{O}_{\mathbb{C}\times\Delta}\left(\sum_{j\geq 1} r^{(\alpha_{ij}-[\alpha_{i1}+a])Id - \frac{N}{2\pi i}} es(A_{ij},\alpha_{ij})\right)$$

so that

$$\mathcal{G}_a \cong Gr_a({}_*\mathcal{E}) = \bigoplus_{i:\alpha_{i1}+a\in\mathbb{Z}} \mathcal{O}_\mathbb{C} r^{-aId-\frac{N}{2\pi i}} es(A_{i1},\alpha_{i1})$$

If $\alpha_{i1}+a\in\mathbb{Z}$ then $es(A_{i1},\alpha_{i1}) \in \oplus_{k\in\mathbb{Z}} Gr_V^{k-a}\mathcal{O}(H)$. On the other hand, the composed map

$$(\mathcal{G}_a)_{|\mathbb{C}^*} \longrightarrow Gr_\mathcal{V}^{-a} \longrightarrow \mathcal{H}^\infty_{e^{2\pi ia}} \longrightarrow \mathcal{O}(H'_{e^{2\pi ia}})$$

sends $r^{-aId-\frac{N}{2\pi i}} es(A_{i1},\alpha_{i1})$ to $es(A_{i1},\alpha_{i1})$ which shows the first part of the lemma. The statement on $\overline{\mathcal{G}}_{-a}$ is proved in the same way. □

The twistor $S^{can}_{(a,0)}(E)$ is constructed in [Moc07] by patching $\mathcal{G}_a$ and $\overline{\mathcal{G}}_{-a}$ via their identifications with $\mathcal{H}^\infty_{e^{2\pi ia}}$ on $\mathbb{C}^*$. Therefore we obtain the following result which shows the first part of theorem 6.7.

Corollary 6.15. *Let $(H, H'_\mathbb{R}, \nabla, P, w)$ be a TERP-structure inducing a nilpotent orbit and $E = p_*\mathcal{C}^{\infty h}(\widehat{\pi^* H})$ the corresponding harmonic bundle. Then the $\mathbb{P}^1$-bundles $\widehat{H^{el}}$ and $\oplus_{a\in L} S^{can}_{(a,0)}(E)$ are isomorphic.*

It remains to identify $(\widehat{W}_\bullet, \widehat{S}, \widehat{N})$ on $\mathcal{O}(\widehat{H^{el}})$ with $(\widehat{\mathcal{W}}_\bullet, \widehat{\mathcal{S}}, 2\pi\cdot\widehat{\mathcal{N}})$ on $\oplus_a S^{can}_{(a,0)}(E)$. The filtrations $\widehat{W}_\bullet$ and $\widehat{\mathcal{W}}_\bullet$ are defined as usual by the nilpotent morphisms $\widehat{N}$ and $\widehat{\mathcal{N}}$, so that it suffices to show the identification of $\widehat{S}$ with $\widehat{\mathcal{S}}$ and of $\widehat{N}$ with $2\pi\cdot\widehat{\mathcal{N}}$.
In order to compare the constructions of the parings and the nilpotent morphisms, remember that Mochizuki starts from a general VPTS (called $(\mathcal{E}^\Delta, \mathbb{D}^\Delta, h)$ on Δ^* in [Moc07, Corollary 11.8]) which is in our case constructed from the variation of TERP-structures $(G, \nabla, G'_\mathbb{R}, P, w)$. In particular, the pairing $\widehat{S}$ on $\widehat{G}$ is precisely the one from lemma 3.9. The pairing $\widehat{\mathcal{S}}$ on $\oplus_a S^{can}_{(a,0)}(E)$ is defined in [Moc07, 11.3.5] through a series on lemmas and intermediate computations. If one follows these definitions carefully, it becomes clear that the isomorphism $\oplus_a S^{can}_{(a,0)}(E) \cong \mathcal{O}(\widehat{H^{el}})$ identifies $\widehat{\mathcal{S}}$ with $\widehat{S}$. The map $\widehat{\mathcal{N}}$ is defined on $\oplus_a\mathcal{G}_a$ resp. $\oplus_a\overline{\mathcal{G}}_{-a}$ by the morphism $\frac{1}{iz}\mathcal{N}_E$ resp. by $iz\mathcal{N}_{\overline{E}}$ [Moc07, 11.3.6] with $\mathcal{N}_E$ from 6.9. This implies that $\widehat{\mathcal{N}} : \oplus_a S^{can}_{(a,0)}(E) \to S^{can}_{(a,0)}(E)\otimes\mathcal{O}_{\mathbb{P}^1}(2)$. Under the isomorphism $\oplus_a\mathcal{G}_a \cong \mathcal{O}(H^{el})$ resp. $\oplus_a\overline{\mathcal{G}}_{-a} \cong \mathcal{O}(\widehat{H}^{el}_{|\mathbb{P}^1\setminus\{0\}})$, $\frac{1}{z}\mathcal{N}_E$ corresponds to $-\frac{1}{2\pi i}\widehat{N}_{|\mathbb{C}}$ and $z\mathcal{N}_{\overline{E}}$ to $\frac{1}{2\pi i}\widehat{N}_{|\mathbb{P}^1\setminus\{0\}}$. Therefore, $\widehat{\mathcal{N}}$ on $\oplus_a S^{can}_{(a,0)}(E)$ corresponds precisely to $\frac{1}{2\pi}\widehat{N}$ on $\mathcal{O}(\widehat{H}^{el})$.

Remark: In [Moc07] there is another construction of a limit mixed twistor structure which uses the same bundles $\mathcal{G}_a$ and $\overline{\mathcal{G}}_{-a}$ as above but a different gluing. This gluing procedure depends this time on a point x "near the origin in Δ^*" and produces a $\mathbb{P}^1$-bundle denoted by $S_{(a,0)}(E,x)$. Although we do not need this construction in order to prove theorem 6.6, we will give here the corresponding statement for the case of nilpotent orbits of TERP-structures.

Lemma 6.16. *Consider the "rescaled" TERP-structure* $\pi_x^*(H, H'_{\mathbb{R}}, P, w)$*, then the corresponding twistor* $\widehat{(\pi_x^* H)^{el}}$ *is isomorphic to* $\bigoplus_{0<a\leq 1} S_{(a,0)}(E,x)$ *and under this isomorphism* $\widehat{W}_\bullet, \widehat{S}$ *get identified with* $\widehat{\mathcal{W}}_\bullet, \widehat{\mathcal{S}}$ *and* $\widehat{N}$ *is mapped to* $2\pi \cdot \widehat{\mathcal{N}}$.

Proof. The gluing of $\mathcal{G}_a$ and $\overline{\mathcal{G}}_{-a}$ to $S_{(a,0)}(E,x)$ is done via an identification

$$(\mathcal{G}_a)_{|\mathbb{C}^*} \cong \mathcal{G}_a(\mathcal{E})_{|\mathbb{C}^*\times\{0\}} \xrightarrow[\Phi_x]{\cong} \overline{\mathcal{G}}_{-a}(\overline{\mathcal{E}})_{|\mathbb{C}^*\times\{0\}} \cong (\overline{\mathcal{G}}_{-a})_{|\mathbb{C}^*}$$

where $\mathcal{G}_a(\mathcal{E})$ resp. $\mathcal{G}_{-a}(\overline{\mathcal{E}})$ are locally free sheaves over $\mathcal{O}_{\mathbb{C}^*\times\Delta}$ resp. $\mathcal{O}_{\mathbb{C}^*\times\overline{\Delta}}$, defined in [Moc07, sections 10.1.4 and 10.3.2]. In our situation, the fact that the filtration $\mathcal{F}_\bullet$ on $\mathcal{H}^\infty$ is trivial shows that

$$\mathcal{G}_a(\mathcal{E}) \cong \left\{ s \in i_*\mathcal{O}(G'_{e^{2\pi i a}}) \,|\, |s|_{p^*h} \in O(|r|^{-a-\epsilon})\, \forall \epsilon > 0 \right\}$$

where $i : \mathbb{C}^* \times \Delta^* \hookrightarrow \mathbb{C}^* \times \Delta$ and G'_λ denotes the flat generalized eigensubbundle of G' with respect to either horizontal or vertical monodromy. From the proof of lemma 6.14 it is obvious that $\mathcal{G}_a(\mathcal{E})$ resp. $\overline{\mathcal{G}}_{-a}(\overline{\mathcal{E}})$ are generated by sections of the form $r^{aId-N/2\pi i} es(A, a+k)$ resp. $\overline{r}^{aId+N/2\pi i} es(\overline{A}, -a+l)$ with $k, l \in \mathbb{Z}$. The identification Φ_x is defined in [Moc07, section 11.3.3] as a composition $\Phi_x = {\Phi^\dagger}_{x,0}^{-1} \circ \Phi_{x,0}$ where

$$\Phi_{x,0} : \mathcal{G}_a(\mathcal{E})_{|\mathbb{C}^*\times\{0\}} \cong \mathcal{G}_a(\mathcal{E})_{|\mathbb{C}^*\times\{x\}}$$

$$\Phi^\dagger_{x,0} : \overline{\mathcal{G}}_{-a}(\overline{\mathcal{E}})_{|\mathbb{C}^*\times\{0\}} \cong \overline{\mathcal{G}}_{-a}(\overline{\mathcal{E}})_{|\mathbb{C}^*\times\{x\}}$$

are isomorphisms of fibres which in our situation boil down to restricting the sections of the above type to $\mathbb{C} \times \{x\}$. The isomorphism $\mathcal{G}_a(\mathcal{E})_{|\mathbb{C}^*\times\{x\}} \cong \overline{\mathcal{G}}_{-a}(\overline{\mathcal{E}})_{|\mathbb{C}^*\times\{x\}}$ in [Moc07] is then simply the identification $\tau : (\pi_x^* H)^{el} \cong \gamma^*(\overline{\pi_x^* H})^{el}$ which shows that $S_{(a,0)}(E,x) \cong \widehat{(\pi_x^* H)^{el}}$. The statements on $\widehat{N}, \widehat{S}, \widehat{\mathcal{N}}$ and $\widehat{\mathcal{S}}$ are proved as in theorem 6.6. □

7 Sabbah's mixed Hodge structures

The results of the previous chapter strongly rely on the assumption of regularity of the given TERP-structure. In the general case, one can no longer define a filtration on the space H^∞ as in corollary 6.4. However, there exists another procedure due to Sabbah ([Sab]) which applies to arbitrary bundles with meromorphic connections. It was used to construct MHS for tame functions, see chapter 11. It uses global sections with moderate growth at infinity. We recall it briefly below. We will get a statement similar to theorem 6.6 but this time PHMS will correspond to Sabbah orbits instead of nilpotent orbits (which explains their name).
Consider an arbitrary TERP-structure $(H, \nabla, H'_{\mathbb{R}}, P, w)$. In the spirit of the beginning of chapter 6, we will consider the Deligne-extensions of H but this time over infinity. More precisely, denote $\widetilde{i} : \mathbb{C} \hookrightarrow \mathbb{P}^1$ and put

$$\widetilde{i}_*\mathcal{O}(H) \supset \mathcal{O}(H_{<\infty}) := \{ s \in \mathcal{O}(H) \,|\, s \text{ has moderate growth at } \infty \}$$

Then $H_{<\infty}$ is a bundle on $\mathbb{P}^1$, meromorphic at infinity (i.e., a locally free $\mathcal{O}_{\mathbb{P}^1}(*\infty)$-module). The rational bundle corresponding to it by GAGA is denoted by G_0. It is a free $\mathbb{C}[z]$-module. It is a lattice at zero inside of $G := G_0 \otimes \mathbb{C}[z, z^{-1}]$, which is rational with poles at zero and infinity. Moreover, for any $\alpha \in \mathbb{C}$ we can consider the following lattices at infinity

$$V_\alpha^{Sab} := \left\{ \omega \in G \,\middle|\, \omega = \textstyle\sum_{\beta\leq\alpha} s(\omega, \beta) \in C^\beta \right\},$$

$$V_{<\alpha}^{Sab} := \left\{ \omega \in G \,\middle|\, \omega = \textstyle\sum_{\beta<\alpha} s(\omega, \beta) \in C^\beta \right\}.$$

Due to the formula $z^{-1}\nabla_{\partial_{z^{-1}}} = -z\nabla_{\partial_z}$, it does not matter whether we consider elementary sections at zero or at infinity. Therefore the above formulas defines the Deligne lattices at infinity of $H_{<\infty}$. V_α^{Sab} and $V_{<\alpha}^{Sab}$ are free $\mathbb{C}[z^{-1}]$-modules inside of G of maximal rank corresponding to algebraic bundles on $\mathbb{P}^1\backslash\{0\}$. The advantage of working algebraically is that we can use the V-filtration at infinity to define a TERP-structure generated by elementary sections. Namely, we write $Gr_\alpha^{V^{Sab}} := V_\alpha^{Sab}/V_{<\alpha}^{Sab} \cong C^\alpha$ as before and use the filtration induced by the V_α^{Sab}'s on G_0.

Definition 7.1. *Let $(H, \nabla, H'_{\mathbb{R}}, P, w)$ be any TERP-structure. Define*

$$G_0^{Sab} = \oplus_\alpha \mathbb{C}[z] Gr_a^{V^{Sab}} G_0$$

as the algebraic bundle generated by the Sabbah-principal parts of sections if G_0. Let H^{Sab} be its analytic counterpart, i.e., $\mathcal{O}(H^{Sab}) = \oplus \mathcal{O}_{\mathbb{C}} Gr_a^{V^{Sab}} G_0$. The spectrum in this situation is defined as

$$Sp^{Sab}(H, \nabla) = \sum_{\alpha\in\mathbb{C}} \nu(\alpha)\alpha \in \mathbb{Z}[\mathbb{C}] \quad \textit{with} \quad \nu(\alpha) := \dim_{\mathbb{C}} \left(\frac{Gr_\alpha^{V^{Sab}} G_0}{Gr_\alpha^{V^{Sab}} z G_0} \right)$$

Write as before the spectral numbers as an ordered $rank(H)$-tuple of (possibly repeated) numbers $\alpha_1 \le ... \le \alpha_{rank(H)}$. We call them spectral numbers at infinity.

The following result is the analogue of lemma 6.3.

Lemma 7.2. *Let $(H, H'_{\mathbb{R}}, \nabla, P, w)$ be a TERP-structure.*

1. *The spectral numbers derived from the V-filtration at infinity satisfy the symmetry $\alpha_i + \alpha_{rank(H)+1-i} = w$.*
2. *$(H^{Sab}, H'_{\mathbb{R}}, \nabla, P, w)$ is again a TERP-structure. It is generated by elementary sections and thus regular singular.*

Proof. $(\mathcal{O}(H^{Sab}), \nabla)$ has a pole of order at most two at zero by the same proof as in lemma 6.3. Moreover, it is obvious that the pairing P induces a pairing $G \otimes j^*G \to \mathbb{C}[z, z^{-1}]$ sending G_0 to $z^w\mathbb{C}[z]$. Then we are exactly in the situation considered in [Sab]. It is shown in section 3 of loc.cit. that the spectral numbers are symmetric and the properties of P as a pairing on G_0^{Sab} we are after are a direct consequence. □

As in the regular singular case, the last lemma allows us to define a filtration on the space H^∞: The TERP-structure $(H^{Sab}, H'_{\mathbb{R}}, \nabla, P, w)$ is generated by elementary sections and is therefore equivalent to the data considered in lemma 5.7. In particular, the formula

$$F^p_{Sab} H^\infty_{e^{-2\pi i\alpha}} := \psi_\alpha^{-1} z^{p+1-w} Gr^{V^{Sab}}_{\alpha+w-1-p} G_0 \cong \psi_\alpha^{-1} \left(z^{p+1-w} (C^{\alpha+w-1-p} \cap \mathcal{O}(H^{Sab})_0) \right) \tag{7.1}$$

for $\alpha \in (0,1] + i\mathbb{R}$ defines a decreasing filtration on H^∞ such that its twisted version $\widetilde{F}^\bullet_{Sab} H^\infty := G^{-1}(F^\bullet_{Sab} H^\infty)$ satisfies the orthogonality conditions (5.8) and therefore gives an element of the classifying space $\check{D}$.
The next result is the "Sabbah-orbit" version of theorem 6.6.

Theorem 7.3. *For an arbitrary TERP-structure $(H, H'_{\mathbb{R}}, \nabla, P, w)$, the following conditions are equivalent.*

1. *$(H, H'_{\mathbb{R}}, \nabla, P, w)$ induces a Sabbah orbit.*
2. *$(H^\infty, H^\infty_{\mathbb{R}}, N, S, \widetilde{F}^\bullet_{Sab})$ defines a PMHS of weight $w-1$ resp. w on $H^\infty_{\arg\neq 0}$ resp. $H^\infty_{\arg=0}$.*

Proof. The proof of 2) —¿ 1) is similar to the proof of [Her03, theorem 7.20]. The parts (I)–(III) in that proof are unchanged, the parts (IV) and (V) can be adapted easily.
The proof of 1)—¿2) is virtually the same as the one in chapter 6. We give only comments on the necessary adjustments. The first point is that by lemma 4.6, if we consider the variation of pure polarized twistor structures associated to the orbit $K := (\pi')^*H$, then taking global sections along the projection $p : \mathbb{P}^1 \times \Delta^* \to \Delta^*$ gives a harmonic bundle E on Δ^* with tame behavior at $0 \in \Delta$.
The discussion of $\mathcal{U}_{|r}$ in the proof of lemma 4.6 shows that on any extension ${}_a\mathcal{E}^0$ of the harmonic bundle E the residue at 0 of the Higgs field is nilpotent. Therefore the KMSS-spectrum of the variation of twistor structures is precisely as described in lemma 6.11. All the properties of the extension sheaves ${}_a\mathcal{E}$ remain true. The vertical and horizontal monodromy are not equal, but inverse to each other. We continue to denote by $\mathcal{H}^\infty$ the flat bundle $\psi_r((K')^\nabla)$, and by $\mathcal{H}^\infty_\lambda$ the eigenbundle of the *horizontal* monodromy. With this convention, the definition of the extensions $\mathcal{V}^\alpha$ remains unchanged. However, we see that now the bundle $\mathcal{O}(K)$ is generated by sections of the form

$$\sum_{i\geq 1} r^{-\alpha_{ij} Id + \frac{N}{2\pi i}} es(A_i, \alpha_{ij})$$

where $\sum_{i\geq 1} es(A_{ij},\alpha_{ij})$ for $j=1,\ldots,rank(H)$ is a set of generating sections of $\mathcal{O}(H^{Sab})$ and where this time $\alpha_{1j}>\alpha_{2j}>\ldots$. The remaining part of the proof is exactly the same. We identify the limit polarized mixed twistor structure $(\oplus_a S^{can}_{(a,0)}(E),\widehat{\mathcal{W}}_\bullet,\widehat{\mathcal{N}},\widehat{\mathcal{S}})$ of Mochizuki with the twistor $(\widehat{H^{Sab}},\widehat{W}_\bullet,-\frac{1}{2\pi}\widehat{N},\widehat{S})$ which is generated by elementary sections and corresponds by lemma 5.9 to a sum of two PMHS. Note that the different signs $+N$ instead of $-N$ in the PMHS in 2. and $-\frac{1}{2\pi}$ instead of $\frac{1}{2\pi}$ in the PMTS result from the fact that now horizontal and vertical monodromy are inverse, not equal. □

8 Formal structure and Stokes structure

From now on we will consider TERP-structures which are not necessarily regular singular. This chapter is devoted to discuss both formal and Stokes structures of these objects. Our main references for the facts discussed below are [Mal83] and [Sab02].

Let $(H,\nabla,H'_{\mathbb{R}},P)$ be a TERP-structure. We will forget about the real subbundle $H'_{\mathbb{R}}$ for a moment. The remaining object (H,∇,P) is called a TEP-structure. We might as well consider only the germ $(\mathcal{H}_0,\nabla,P)$ at zero. It turns out that for TEP-structures, it is sufficient to work at the level of formal power series, i.e., to consider only the formalization $(\mathcal{H}_0,\nabla,P)\otimes\mathbb{C}[[z]][z^{-1}]$. However, to incorporate the real subbundle of a TERP-structure, we will need to use the Stokes structure associated to the irregular connection ∇.

To start with, let us simplify the situation even more and restrict our attention to the germ $\mathcal{H}_0[z^{-1}]=\mathcal{H}_0\otimes\mathbb{C}\{z\}[z^{-1}]$. Remember that the connection has a pole of order two on $\mathcal{H}_0[z^{-1}]$. By the theorem of Turrittin (e.g. [Mal83, 2.1]) there is a finite ramification $r_n:\mathbb{C}\to\mathbb{C}$, $z\mapsto z^n$, such that $r_n^*(\mathcal{H}_0[z^{-1}],\nabla)$ is formally isomorphic to a sum $\bigoplus_{i=1}^l e^{f_i}\otimes(\mathcal{R}_i,\nabla_i)$, where $f_i\in\mathbb{C}[z^{-1}]$, $\mathcal{R}_i$ is a $\mathbb{C}\{z\}[z^{-1}]$-vector space and $(\mathcal{R}_i,\nabla_i)$ is regular singular. The symbol e^{f_i} denotes the bundle of rank one equipped with the connection given by the one form df_i (equivalently, $e^{f_i}\otimes\mathcal{R}_i$ is the germ of the meromorphic extension at zero given by multiplying sections in $\mathcal{R}_i$ by e^{f_i}).

In the sequel, we will always make the simplifying assumption that the ramification is unnecessary. This is satisfied in all examples (singularity theory) and potential examples (quantum cohomology) which we have in mind. In this situation, the exponents that occur in the formal decomposition are simply $f_i=-\frac{u_i}{z}$, where u_i is an eigenvalue of the pole part $\mathcal{U}$. This is encapsulated in the following definition.

Definition 8.1. *A TEP-structure (or its formalization at zero) (H,∇,P) is said to require no ramification iff the germ $(\mathcal{H}_0[z^{-1}],\nabla)$ is formally isomorphic to a sum $\bigoplus_{i=1}^l e^{-u_i/z}\otimes(\mathcal{R}_i,\nabla_i)$ where $\mathcal{R}_i$ is a $\mathbb{C}\{z\}[z^{-1}]$-vector space, $(\mathcal{R}_i,\nabla_i)$ is regular singular, and $u_1,...,u_l$ are the different eigenvalues of the pole part $\mathcal{U}=[z\nabla_{z\partial_z}]$. A TERP-structure is said to require no ramification iff this is true for the corresponding TEP-structure.*

Given a TEP-structure $(\mathcal{H}_0,\nabla,P)$, then the $\mathbb{C}\{z\}$-lattice $e^{-u/z}\cdot\mathcal{H}_0$ with the induced connection (in the above sense) and the induced pairing is also a TEP structure.

Lemma 8.2. *Let $(\mathcal{H}_0,\nabla,P)$ be a TEP-structure which does not require a ramification and Ψ a formal isomorphism from $(\mathcal{H}_0[z^{-1}],\nabla)$ to a sum $\bigoplus_{i=1}^l e^{-u_i/z}\otimes(\mathcal{R}_i,\nabla_i)$ as above. Then there are unique $\mathbb{C}\{z\}$-lattices $(\mathcal{H}_i)_0\subset\mathcal{R}_i$ and pairings P_i such that $((\mathcal{H}_i)_0,\nabla_i,P_i)$ is a regular singular TEP-structure and Ψ is an isomorphism of the formal TEP-structures $(\mathcal{H}_0\otimes\mathbb{C}[[z]],\nabla,P)$ and $\bigoplus_{i=1}^l e^{-u_i/z}\otimes((\mathcal{H}_i)_0\otimes\mathbb{C}[[z]],\nabla_i,P_i)$. The TEP-structures $((\mathcal{H}_i)_0,\nabla_i,P_i)$ are called the regular singular pieces of $(\mathcal{H}_0,\nabla,P)$.*

Proof. Exercise 5.9 in [Sab02, II] shows that the $\mathbb{C}[[z]]$-lattice $\Psi(\mathcal{H}_0\otimes\mathbb{C}[[z]])\subset\Psi(\mathcal{H}_0\otimes\mathbb{C}[[z]][z^{-1}])$ is compatible with the splitting $\Psi(\mathcal{H}_0\otimes\mathbb{C}[[z]][z^{-1}])=\bigoplus_{i=1}^l e^{-u_i/z}\otimes(\mathcal{R}_i\otimes\mathbb{C}[[z]][z^{-1}])$, that means, it splits into l $\mathbb{C}[[z]]$-lattices. Because of [Sab02, III.2.1] these l $\mathbb{C}[[z]]$-lattices arise as $e^{-u_i/z}\otimes((\mathcal{H}_i)_0\otimes\mathbb{C}[[z]])$ from some $\mathbb{C}\{z\}$-lattices $(\mathcal{H}_i)_0\subset\mathcal{R}_i$.

The connection on $\mathcal{R}_i$ has a pole of order ≤ 2 with respect to $(\mathcal{H}_i)_0$, because the same holds for the connection on the factor $e^{-u_i/z}$ and for the connection on $e^{-u_i/z}\otimes(\mathcal{R}_i\otimes\mathbb{C}[[z]][z^{-1}])$ with respect to the $\mathbb{C}[[z]$-lattice $e^{-u_i/z}\otimes((\mathcal{H}_i)_0\otimes\mathbb{C}[[z]])$.

The pairing $\Psi(P)$ on $\Psi(\mathcal{H}_0\otimes\mathbb{C}[[z]])$ induced by P gives the first of the following isomorphisms (compare the

proof of lemma 6.3).

$$
\begin{aligned}
(z^{-w}\Psi(\mathcal{H}_0\otimes\mathbb{C}[[z]]),\nabla) &\cong j^*(\Psi(\mathcal{H}_0\otimes\mathbb{C}[[z]])^*,\nabla^*) \\
&\cong j^*\left(\bigoplus_{i=1}^{l} e^{u_i/z}\otimes((\mathcal{H}_i)_0^*,\nabla_i^*)\otimes\mathbb{C}[[z]]\right) \\
&\cong \bigoplus_{i=1}^{l} e^{-u_i/z}\otimes j^*((\mathcal{H}_i)_0,\nabla_i^*)\otimes\mathbb{C}[[z]]. \qquad (8.1)
\end{aligned}
$$

As the formal decomposition is unique (e.g. [Sab02, II.5.5]), these isomorphisms respect the decomposition and give rise to isomorphisms $(z^{-w}(\mathcal{H}_i)_0,\nabla_i)\cong j^*((\mathcal{H}_i)_0^*,\nabla_i^*)$ corresponding to pairings P_i on $(\mathcal{H}_i)_0$ which make $((\mathcal{H}_i)_0,\nabla_i,P_i)$ into regular singular TEP-structures. Moreover, for $i\neq j$, the pairing $\Psi(P)$ vanishes on $e^{-u_i/z}\otimes(\mathcal{H}_i)_0\times e^{-u_j/z}\otimes(\mathcal{H}_j)_0$. $\square$

As remarked above, the formal structure of the connection $(\mathcal{H}_0,\nabla)$ is not sufficient to treat the real subbundle $H'_\mathbb{R}$. The necessary analytic information is provided by what is called the Stokes structure associated to $(\mathcal{H},\nabla)$. Roughly speaking, it consists of data which keep track of the difference of analytic liftings of the formal decompositions in sectors of $\mathbb{C}$. To discuss this Stokes structure, we need some notations.
For any $I\subset S^1$, let $\widehat{I}:=\{z\in\mathbb{C}^*\mid \frac{z}{|z|}\in I\}$. If $I\subset S^1$ happens to be open and connected, then $\widehat{I}$ is a sector. Denote by $\mathcal{A}[z^{-1}]$ the sheaf on S^1 of holomorphic functions in neighborhoods of 0 in sectors which have an asymptotic development in the sense of [Mal83, 3.]. For any $\xi\in S^1$ the Taylor development of functions in $\mathcal{A}[z^{-1}]_\xi$ yields a map $T:\mathcal{A}[z^{-1}]_\xi\to\mathbb{C}[[z]][z^{-1}]$ which is surjective by the lemma of Borel-Ritt. We will need the subsheaves $\mathcal{A}:=T^{-1}(\mathbb{C}[[z]])$ and $\mathcal{A}^{<0}:=\ker T$ of $\mathcal{A}[z^{-1}]$. Remark that $\gamma^*(\overline{\mathcal{A}[z^{-1}]})$ is the sheaf on S^1 of holomorphic functions in neighborhoods of ∞ in sectors such that the functions have asymptotic developments at ∞.
Fix a TERP-structure $(H,\nabla,H'_\mathbb{R},P)$ which does not require a ramification and a formal isomorphism

$$
\Psi:(\mathcal{H}_0,\nabla,P)\otimes\mathbb{C}[[z]]\overset{\cong}{\to}\bigoplus_{i=1}^{l} e^{-u_i/z}\otimes((\mathcal{H}_i)_0,\nabla_i,P_i)\otimes\mathbb{C}[[z]]
$$

of TEP-structures. In this situation, we say that $\xi\in S^1$ is a Stokes direction if there exist $i\neq j$ with $\Re(\frac{u_i-u_j}{\xi})=0$. These Stokes directions form a finite set $\Sigma\subset S^1$. For $\xi\in S^1\backslash\Sigma$, the component of $S^1-\Sigma$ which contains ξ is denoted by $I(\xi)$ and we define an order on $\{1,...,l\}$ depending on ξ as follows

$$
\begin{aligned}
i\leq_\xi j \quad &\overset{\text{def}}{\iff}\quad \Re\left(\frac{u_i}{\xi}\right)<\Re\left(\frac{u_j}{\xi}\right)\quad\text{or}\quad i=j \qquad (8.2)\\
&\iff\quad (z\mapsto e^{\frac{u_i-u_j}{z}})\in\mathcal{A}_\xi \iff (z\mapsto e^{(\overline{u_i}-\overline{u_j})z})\in\gamma^*(\overline{\mathcal{A}_\xi}).
\end{aligned}
$$

In the sequel, we will work with a covering of S^1 by two open sets $I_\pm(\xi)$ defined in the following way: Choose once and for all a value $\xi\in S^1\backslash\Sigma$ and put

$$
I_\pm(\xi):=I(\xi)\cup I(-\xi)\cup\{z\in S^1\mid \pm\Im(z/\xi)\leq 0\}.
$$

Each of the sets $I_+(\xi)$ and $I_-(\xi)$ contains exactly one of two Stokes directions $\pm\xi'$ for any $\xi'\in\Sigma$. Let $\mathcal{L}$ be the local system of $(H,\nabla)_{|S^1}$, $\mathcal{L}_\mathbb{R}\subset\mathcal{L}$ the local system of $(H'_\mathbb{R},\nabla)_{|S^1}$ and denote by $\mathcal{L}_i$ the local system of flat sections of the i-th regular singular piece H_i. The theorem of Hukuhara (e.g. [Mal83, 3.] or [Sab02, II.5.12]) and the discussion in [Mal83, 4.+5.] yield the following result.

Lemma 8.3. *Let $(H,\nabla,H'_\mathbb{R},P)$ be a TERP-structure requiring no ramification and let Ψ and $\bigoplus_{i=1}^{l} e^{-u_i/z}\otimes((\mathcal{H}_i)_0,\nabla_i,P_i)$ and $\xi\in S^1$ be as above. There is a unique lift $\Psi_\pm$ of Ψ to an isomorphism of sheaves on $I_\pm(\xi)$,*

$$
\Psi_\pm:\mathcal{A}[z^{-1}]_{|I_\pm(\xi)}\otimes(\mathcal{H}_0[z^{-1}],\nabla)\longrightarrow\mathcal{A}[z^{-1}]_{|I_\pm(\xi)}\otimes\left(\bigoplus_{i=1}^{l} e^{-u_i/z}\otimes((\mathcal{H}_i)_0[z^{-1}],\nabla_i)\right). \qquad (8.3)
$$

The underlying isomorphism of local systems on $I_\pm(\xi)$ is denoted by the same letter

$$\Psi_\pm : \mathcal{L}_{|I_\pm(\xi)} \to \bigoplus_{i=1}^{l} \mathcal{L}_{i|I_\pm(\xi)}.$$

Let $\mathcal{L}_{\pm,i} := \Psi_\pm^{-1}(\mathcal{L}_{i|I_\pm(\xi)})$, so that $\mathcal{L}_{|I_\pm(\xi)} = \bigoplus_{i=1}^{l} \mathcal{L}_{\pm,i}$. At $\pm\xi$ the two splittings induce (restrictions of) projections $t_{ij}^{(\pm)} : (\mathcal{L}_{+,i})_{\pm\xi} \to (\mathcal{L}_{-,j})_{\pm\xi}$ satisfying

$$t_{ij}^{(+)} = 0 \quad \Longleftarrow \quad i <_\xi j \quad \Longleftrightarrow \quad j <_{-\xi} i \quad \Longrightarrow \quad t_{ji}^{(-)} = 0.$$

Therefore $t_{ii}^{(\pm)}$ are isomorphisms, and the local system obtained by gluing $\mathcal{L}_{+,i}$ and $\mathcal{L}_{-,i}$ with $t_{ii}^{(\pm)}$ is mapped by $\Psi_\pm$ to $\mathcal{L}_i$.

The following lemma extends the decomposition of the meromorphic bundle to the lattice $\mathcal{H}_0$ and describes the behavior of the pairing.

Lemma 8.4. *We consider the same situation as in the last lemma. The formal isomorphism Ψ respects the $\mathbb{C}[[z]]$-lattice $\mathcal{H}_0 \otimes \mathbb{C}[[z]]$ and thus induces isomorphisms*

$$\Psi_\pm : \mathcal{A}_{|I_\pm(\xi)} \otimes (\mathcal{H}_0, \nabla) \to \mathcal{A}_{|I_\pm(\xi)} \otimes \left(\bigoplus_{i=1}^{l} e^{-u_i/z} \otimes ((\mathcal{H}_i)_0, \nabla_i) \right). \tag{8.4}$$

The two splittings $\bigoplus_{i=1}^{l} \mathcal{L}_{\pm,i}$ of $\mathcal{L}_{|I_\pm(\xi)}$ are dual with respect to the pairing P. The maps $t_{ij}^{(+)}$ and $t_{ji}^{(-)}$ determine each other by

$$P(t_{ij}^{(+)}-,-) = P(-,-) = P(-,t_{ji}^{(-)}-) : (\mathcal{L}_{+,i})_\xi \times (\mathcal{L}_{+,j})_{-\xi} \longrightarrow \mathbb{C}. \tag{8.5}$$

P restricts to the local system obtained by gluing $\mathcal{L}_{\pm,i}$ with $t_{ii}^{(\pm)}$. It is identified with P_i by the isomorphism with $\mathcal{L}_i$.

Proof. The first statement is clear. For the second point, suppose $i \neq j$, $\sigma_i \in (\mathcal{H}_i)_0$ and $\sigma_j \in (\mathcal{H}_j)_0$. The last statement in the proof of lemma 8.2 shows that $P\left(\Psi_+^{-1}(e^{-u_i/z}\sigma_i), \Psi_-^{-1}(e^{-u_j/z}\sigma_j)\right)$ is an element in $\Gamma(I_+(\xi), \mathcal{A}^{<0})$. On the other hand, it takes the form $e^{(u_j-u_i)/z} \cdot \sum_{\alpha \geq \alpha_0, k} a_{\alpha,k} z^\alpha (\log z)^k$. Therefore it vanishes. This shows the duality property of the two splittings and also yields both equalities in (8.5). The remaining statements are obvious. □

The next definition introduces the condition we need in order to equip the regular singular pieces with a real structure, i.e., make them into TERP-structures.

Definition 8.5. *Let $(H, \nabla, H'_\mathbb{R}, P)$ be a TERP-structure requiring no ramification. Consider the data from lemma 8.3. We say that real structure and Stokes structure are compatible if $\mathcal{L}_{\pm,i} = \overline{\mathcal{L}_{\pm,i}}$.*

Lemma 8.6. *Let $(H, \nabla, H'_\mathbb{R}, P)$ be a TERP-structure requiring no ramification and which has compatible real structure and Stokes structure. Then the regular singular pieces (H_i, ∇_i, P_i), which are TEP-structures by lemma 8.2, are naturally equipped with real structures and become TERP-structures.*

Proof. The identity map on the stalks $\mathcal{L}_\pm$ can be decomposed as $Id = \sum_{j \leq_{\pm\xi} i} t_{ij}^{(\pm)} : \mathcal{L}_{\pm\xi} \to \mathcal{L}_{\pm\xi}$. As the identity obviously respects the real structure, the same holds true for the individual maps $t_{ij}^{(\pm)} : (\mathcal{L}_{+,i})_{\pm\xi} \to (\mathcal{L}_{-,j})_{\pm\xi}$. By lemma 8.3, $\mathcal{L}_i$ is canonically isomorphic to the local system obtained by gluing $\mathcal{L}_{\pm,i}$ with $t_{ii}^{(\pm)}$. Therefore it carries a canonical real structure. P_i maps the real local system in $\mathcal{L}_i$ to $i^w\mathbb{R}$ because of the identification in lemma 8.4 with the restriction of P. This gives a real flat subbundle $H'_{i,\mathbb{R}} \subset H'_i$ such that $(H_i, \nabla_i, H'_{i,\mathbb{R}}, P_i)$ is a TERP-structure. □

The following lemma is now an immediate consequence of the preceding results.

Lemma 8.7. *Let $(H, \nabla, H'_{\mathbb{R}}, P)$ be a TERP-structure requiring no ramification. The integrable twistor $(\widehat{\mathcal{H}}, \nabla)$ has a pole of order two at infinity and the map $\tau \circ \Psi_\pm \circ \tau$ is an isomorphism of sheaves on $I_\pm(\xi)$:*

$$\tau \circ \Psi_\pm \circ \tau : \gamma^* \left(\overline{\mathcal{A}}_{|I_\pm(\xi)}\right) \otimes (\widehat{\mathcal{H}}_\infty, \nabla) \longrightarrow \gamma^* \left(\overline{\mathcal{A}}_{|I_\pm(\xi)}\right) \otimes \left(\bigoplus_{i=1}^{l} e^{-\overline{u_i} \cdot z} \otimes ((\widehat{\mathcal{H}_i})_\infty, \nabla_i) \right). \tag{8.6}$$

It is the unique lift on $I_\pm(\xi)$ of the corresponding formal isomorphism. On the level of local systems it is given by the composition $\overline{} \circ \Psi_\pm \circ \overline{} =: \overline{\Psi_\pm}$. Therefore it induces the splitting $\mathcal{L}_{|I_\pm(\xi)} = \bigoplus_{i=1}^l \overline{\mathcal{L}_{\pm,i}}$. The morphisms $\Psi_\pm$ and $\tau \circ \Psi_\pm \circ \tau$ glue to an isomorphism of sheaves on the real blow up $[0, \infty] \times I_\pm(\xi)$ at 0 and ∞ of the sector $\widehat{I}_\pm(\xi)$ iff real structure and Stokes structure are compatible.

9 Mixed TERP-structures

The correspondence between nilpotent orbits of Hodge structures and PHMS due to Cattani, Kaplan and Schmid (theorem 2.5) was generalized to regular singular TERP-structures in chapter 6 (theorem 6.6). In conjecture 9.2 we propose a further generalization to arbitrary TERP-structures. We will give a complete proof of one direction of this correspondence. In order to state the general correspondence, we need replace the notion of PHMS by what we call mixed TERP-structure. The definition is straightforward after what has been said in the last chapter.

Definition 9.1. *A TERP-structure is a mixed TERP-structure if it does not require a ramification, if real structure and Stokes structure are compatible and if the regular singular pieces of lemma 8.2, which are TERP-structures by lemma 8.6, induces PMHS as in theorem 6.6.*

With this definition at hand, the main conjecture is very simple to state, and takes precisely the same form as in the regular singular case of chapter 6.

Conjecture 9.2. *A TERP-structure which does not require a ramification is a mixed TERP-structure if and only if it induces a nilpotent orbit.*

The following is the main result of this paper.

Theorem 9.3. *1. The conjecture is true if the TERP-structure is regular singular.*

2. The implication $\Rightarrow$ is true.

As already said, the first statement is precisely theorem 6.6. The remaining part of this chapter is concerned with a proof of the second claim.
Part 2. is already known in two cases which are opposite to each other in a certain sense: If the TERP-structure is regular singular, i.e., if $\mathcal{U}$ is nilpotent, it is precisely [Her03, theorem 7.20]. If all eigenvalues of $\mathcal{U}$ are different (we call the TERP-structure semi-simple in that case), then it follows from [Dub93, proposition 2.2]. We will use [Her03, theorem 7.20] and combine it with ideas from [Dub93] to get the general case. This will contain a new proof of the semi-simple case.
Let $(H, \nabla, H'_{\mathbb{R}}, P)$ be a mixed TERP-structure of weight w. We use all notations and objects from chapter 8. In particular, $\xi \in S^1 \backslash \Sigma$ is chosen, giving rise to the covering $S^1 = I_+(\xi) \cup I_-(\xi)$ with $I_+(\xi) \cap I_-(\xi) = I(\xi) \cup (-I(\xi))$. $\mathcal{L}$ is the local system of $(H, \nabla)_{|S^1}$. There are the canonical splittings $\mathcal{L}_{|I_\pm(\xi)} = \bigoplus_{i=1}^l \mathcal{L}_{\pm,i}$ which are induced from the unique isomorphisms $\Psi_\pm : \mathcal{L}_{|I_\pm(\xi)} \to \bigoplus_{i=1}^l \mathcal{L}_{i|I_\pm(\xi)}$ in (8.4). We choose the numbering $1, \ldots, l$ of the block decomposition in such a way that for all $i, j \in \{1, \ldots, l\}$ we have $i \leq j \iff i \leq_\xi j$. By assumption, the regular singular pieces $(H_i, \nabla_i, H'_{i,\mathbb{R}}, P_i)$ are TERP-structures which induce PMHS as in theorem 6.6. Also lemma 8.7 applies.
Denote $n := rank(H)$ and $n_i := rank(H_i)$. Let $\underline{e}^p = (e_1^\pm, ..., e_n^\pm)$ be bases of $\mathcal{L}_{|I_\pm(\xi)}$ satisfying $P(e_i^+, e_j^-) = (-i)^w \delta_{ij}$ and which are adapted to the splittings, i.e., $(e_1^\pm, ..., e_{n_1}^\pm)$ is a basis of $\mathcal{L}_{\pm,1}$ etc. Such bases exist by lemma 8.4. The images $\underline{f}^\pm = (f_1^\pm, ..., f_n^\pm) = (\Psi_\pm(e_1^\pm), ..., \Psi_\pm(e_n^\pm))$ are bases of the local systems $\bigoplus_{i=1}^l \mathcal{L}_{i|I_\pm(\xi)}$, adapted to the splittings. Because of (8.5) there is a unique matrix $T \in GL(n, \mathbb{C})$ with

$$(\underline{e}^-)_\xi = (\underline{e}^+)_\xi \cdot T \quad \text{and} \quad (\underline{e}^-)_{-\xi} = (\underline{e}^+)_{-\xi} \cdot (-1)^w T^{tr}. \tag{9.1}$$

Write $T=(T_{jk})_{j,k=1,...,l}$ with blocks $T_{jk}\in M(n_j\times n_k,\mathbb{C})$. Lemma 8.3 and the chosen ordering of the factors of the decomposition implies that the matrix T is block upper triangular. Denote $T^{model}:=\operatorname{diag}(T_{11},...,T_{ll})$ and $T^{Stokes}:=(T^{model})^{-1}\cdot T$. Then

$$(\underline{f}^-)_\xi=(\underline{f}^+)_\xi\cdot T^{model}\quad\text{ and }\quad(\underline{f}^-)_{-\xi}=(\underline{f}^+)_{-\xi}\cdot(-1)^w(T^{model})^{tr}. \tag{9.2}$$

We want to prove that $\pi_r^*(H,\nabla,H'_\mathbb{R},P)$ is a polarized pure TERP-structure for $|r|\ll 1$. As has been shown in lemma 4.4, the corresponding twistors are isomorphic for r having the same absolute value, so it is sufficient to prove the case where $r\in\mathbb{R}_{>0}$.
The family $\bigcup_{r>0}\pi_r^*(H,\nabla,H'_\mathbb{R},P)$ is isomonodromic. This implies that data $\mathcal{L}$, $\mathcal{L}_{\pm,i}$, $\mathcal{L}_i$, $\underline{e}^\pm$, $\underline{f}^\pm$ defined above for $r=1$ can be identified with the analogous data for any $r>0$. The family has constant Stokes structure (e.g. [Sab02, II.6.c]). Therefore the isomorphisms $\Psi_\pm:\mathcal{L}_{|I_\pm(\xi)}\to\bigoplus_{i=1}^l\mathcal{L}_{i|I_\pm(\xi)}$ of local systems extend for any $r>0$ to isomorphisms

$$\Psi_\pm(r):\mathcal{A}_{|I_\pm(\xi)}\otimes\pi_r^*(\mathcal{H}_0,\nabla)\to\mathcal{A}_{|I_\pm(\xi)}\otimes\left(\bigoplus_{i=1}^l e^{-\frac{u_i}{z\cdot r}}\otimes\pi_r^*((\mathcal{H}_i)_0,\nabla_i)\right). \tag{9.3}$$

As explained in the last chapter, we will use the notation $e^f\otimes$ to denote extensions of sheaves over zero or infinity (or both) twisted by multiplying sections by the function $(z\mapsto e^f)$ on $\mathbb{C}^*$. Obviously $e^{-u_i/(z\cdot r)}\otimes\pi_r^*(H_i,\nabla_i,H'_{i,\mathbb{R}},P_i)$ is a TERP-structure and is equal to $\pi_r^*(e^{-u_i/z}\otimes(H_i,\nabla_i,H'_{i,\mathbb{R}},P_i))$. The extension to ∞ by τ satisfies

$$[\pi_r^*(e^{-\frac{u_i}{z}}\otimes\mathcal{H}_i)]^\wedge=[e^{-\frac{u_i}{z\cdot r}}\otimes\pi_r^*\mathcal{H}_i]^\wedge=e^{-\frac{u_i}{z\cdot r}-\frac{\overline{u_i}\cdot z}{r}}\otimes\widehat{\pi_r^*\mathcal{H}_i}. \tag{9.4}$$

Lemma 8.7 applies and shows that $\Psi_\pm(r)$ and $\tau\circ\Psi_\pm\circ\tau$ glue to an isomorphism

$$\Psi_\pm(r):\widehat{\mathcal{A}}_\pm\otimes\widehat{\pi_r^*\mathcal{H}}\mapsto\widehat{\mathcal{A}}_\pm\otimes\left(\bigoplus_{i=1}^l e^{-\frac{u_i}{z\cdot r}-\frac{\overline{u_i}\cdot z}{r}}\otimes\widehat{\pi_r^*\mathcal{H}_i}\right). \tag{9.5}$$

Here $\widehat{\mathcal{A}}_\pm$ is a sheaf on the real blow up $[0,\infty]\times I_\pm(\xi)$ of $\widehat{I}_\pm(\xi)$ at 0 and ∞. It is the extension of $\mathcal{O}_{\widehat{I}_\pm(\xi)}$ by $\mathcal{A}_{|I_\pm(\xi)}$ at 0 and $\gamma^*(\overline{\mathcal{A}_{|I_\pm(\xi)}})$ at ∞.
By [Her03, theorem 7.20] resp. theorem 6.6 2.⇒1., $\pi_r^*(H_i,\nabla_i,H'_{i,\mathbb{R}},P_i)$ is a polarized pure TERP-structure for small $r>0$. One can choose a basis $\underline{\sigma}(r)=(\sigma_1(r),...,\sigma_n(r))$ of $\bigoplus_{i=1}^l\Gamma(\mathbb{P}^1,\widehat{\pi_r^*\mathcal{H}_i})$ compatible with the splitting. The matrix $P_{mat}(r):=z^{-w}\cdot P((\underline{\sigma})^{tr},\tau(\underline{\sigma}))$ is independent of z, hermitian, positive definite and block diagonal. The entries of the matrices $C^\pm$ defined by

$$\underline{\sigma}_{|\widehat{I}_\pm(\xi)}=\underline{f}^\pm\cdot C^\pm \tag{9.6}$$

take the form $\sum_{\alpha_1\leq\alpha\leq\alpha_n}\sum_{k\in\mathbb{N}\cup\{0\}}a_{\alpha,k}(r)\cdot z^\alpha\cdot(\log z)^k$, where α_1 and α_n are the minimal and the maximal spectral number of the regular singular TERP-structure $\bigoplus_{i=1}^l\pi_r^*(H_i,\nabla_i,H'_{i,\mathbb{R}},P_i)$. Note that theses spectral numbers are real because by assumption the TERP-structure induces PMHS (see the beginning of the proof of lemma 5.9).
The proof of [Her03, theorem 7.20], and more precisely the formulas (7.107), (7.117) and (7.123), provides a family of bases $\underline{\sigma}(r)$ for small $r>0$ such that the coefficients $a_{\alpha,k}(r)$ in all entries of $C^\pm$ and $(C^\pm)^{-1}$ are real analytic and of order $O(|\log r|^N)$ for some $N>0$.
Also $\bigoplus_{i=1}^l e^{-u_i/(z\cdot r)}\otimes\pi_r^*(H_i,\nabla_i,H'_{i,\mathbb{R}},P_i)$ is a polarized pure TERP-structure for small $r>0$. A basis is $\underline{\sigma}(r)\cdot R(r)$ where

$$R(r):=\operatorname{diag}(\exp(-\frac{u_1}{z\cdot r}-\frac{\overline{u_1}\cdot z}{r})\cdot\mathbf{1}_{n_1},..,\exp(-\frac{u_l}{z\cdot r}-\frac{\overline{u_l}\cdot z}{r})\cdot\mathbf{1}_{n_l}). \tag{9.7}$$

The identities $R(r)=\gamma^*(\overline{R(r)})=R(r)^{tr}=R(r,-z)^{-1}$ show the second equality in

$$z^{-w}\cdot P((\underline{\sigma}(r)\cdot R(r))^{tr},\tau(\underline{\sigma}(r)\cdot R(r)))=R(r)^{tr}\cdot P_{mat}(r)\cdot\gamma^*(\overline{R(r,-z)})=P_{mat}(r). \tag{9.8}$$

We want to show that $\pi_r^*(H, \nabla, H'_{\mathbb{R}}, P)$ is a polarized pure TERP-structure for small $r > 0$. Suppose for a moment that this holds and that $\underline{\omega} = (\omega_1, ..., \omega_n)$ is a basis of $\Gamma(\mathbb{P}^1, \widehat{\pi_r^*\mathcal{H}})$. Then

$$\underline{\omega} = \Psi_{\pm}^{-1}(\underline{\sigma} \cdot R) \cdot A^{\pm} = \underline{e}^{\pm} \cdot C^{\pm} \cdot R \cdot A^{\pm} \tag{9.9}$$

on $\widehat{I}_{\pm}(\xi)$, where $A^{\pm}$ are matrices with entries in $\Gamma([0, \infty] \times I_{\pm}(\xi), \widehat{\mathcal{A}}_{\pm})$. Formula (9.1), that is, $(\underline{e}^-)_{|I(\xi)} = (\underline{e}^+)_{|I(\xi)} \cdot T$, shows on the small sector $\widehat{I}(\xi)$

$$\begin{aligned} \underline{\omega}_{|\widehat{I}(\xi)} &= \left(\underline{e}^+ \cdot C^+ \cdot R \cdot A^+\right)_{|\widehat{I}(\xi)} \\ &= \left(\underline{e}^- \cdot C^- \cdot R \cdot A^-\right)_{|\widehat{I}(\xi)} = \left(\underline{e}^+ \cdot T \cdot C^- \cdot R \cdot A^-\right)_{|\widehat{I}(\xi)} . \end{aligned} \tag{9.10}$$

(9.2) and (9.6) give $C^+_{|\widehat{I}(\xi)} = T^{model} \cdot C^-_{|\widehat{I}(\xi)}$. Therefore

$$A^+_{|\widehat{I}(\xi)} \cdot (A^-_{|\widehat{I}(\xi)})^{-1} = \left(R^{-1} \cdot (C^-)^{-1} \cdot T^{Stokes} \cdot C^- \cdot R\right)_{|\widehat{I}(\xi)} . \tag{9.11}$$

Analogously on $-\widehat{I}(\xi) = \widehat{I}(-\xi)$

$$A^+_{|\widehat{I}(-\xi)} \cdot (A^-_{|\widehat{I}(-\xi)})^{-1} = \left(R^{-1} \cdot (C^+)^{-1} \cdot (-1)^w (T^{Stokes})^{tr} \cdot C^+ \cdot R\right)_{|\widehat{I}(-\xi)} . \tag{9.12}$$

Obviously, the bundle $\widehat{\pi_r^*\mathcal{H}}$ is trivial iff there are invertible matrices $A^{\pm}$ with entries in $\Gamma([0, \infty] \times I_{\pm}(\xi), \widehat{\mathcal{A}}_{\pm})$ and satisfying (9.11) and (9.12). The requirement $A^{\pm}(0) = \mathbf{1}_n$ makes them unique. This is a Riemann boundary value problem. We will argue that the matrices on the right hand side of (9.11) and (9.12) are close to $\mathbf{1}_n$ for small $r > 0$ and that therefore the problem has a solution.
The matrix $T^{Stokes} = (T^{model})^{-1} \cdot T$ is block upper triangular with diagonal blocks $T^{Stokes}_{jj} = \mathbf{1}_{n_j}$ for $j = 1, .., l$. The matrices $C^{\pm}$ are block diagonal. Therefore the matrix on the right hand side of (9.11) is also block upper triangular, and the (j, k)-block for $j \leq k$ is

$$(C_j^-)^{-1} \cdot T^{Stokes}_{jk} \cdot C_k^- \cdot \exp\left(\frac{u_j - u_k}{z \cdot r} + \frac{(\overline{u_j} - \overline{u_k}) \cdot z}{r}\right). \tag{9.13}$$

The diagonal block for $j = k$ is $\mathbf{1}_{n_j}$.
The functions $\left(z \mapsto \exp\left(\frac{u_j - u_k}{z \cdot r} + \frac{(\overline{u_j} - \overline{u_k}) \cdot z}{r}\right)\right)$ on $\widehat{I}(\xi)$ for $j < k$ have asymptotic developments equal to zero at the origin and at infinity. Remark that $\Re(\frac{u_j - u_k}{z}) < 0$ and $\Re((\overline{u_j} - \overline{u_k}) \cdot z) < 0$ for $z \in \widehat{I}(\xi)$. Therefore with $r \to 0$ the functions and all their z-derivatives tend to 0 pointwise. The moderate behavior in z and r of the entries of $C^{\pm}$ and $(C^{\pm})^{-1}$ had been discussed above.
We obtain: The restrictions to $\overline{\xi \cdot \mathbb{R}_{>0}} \subset \mathbb{P}^1$ (here $\overline{}$ denotes closure) of all the entries of a (j, k)-block with $j < k$ are real analytic on $\xi \cdot \mathbb{R}_{>0}$ and C^∞ at 0 and ∞; if $r \to 0$, they and all their derivatives tend uniformly to 0.
The same analysis holds for the nondiagonal entries of the right hand side of (9.12). Here one remarks that this matrix is lower triangular and that $z \in \widehat{I}(-\xi)$. One considers the restrictions to $\overline{\xi \cdot \mathbb{R}_{<0}}$.
By a Möbius transformation one can map $\overline{\xi \cdot \mathbb{R}} \subset \mathbb{P}^1$ to S^1. The Birkhoff decomposition in [PS86, (8.1.2)] gives a certain unique decomposition for all C^∞ loops $S^1 \to GL(n, \mathbb{C})$ in an open dense set of the loop space $LGL(n, \mathbb{C})$, which contains 0; and this decomposition depends smoothly on the loop.
In our case it shows the existence and uniqueness of matrices $A^{\pm}$ with the following properties: $A^{\pm}$ is continuous and invertible on the set $\overline{\{z \in \mathbb{C}^* \mid \pm \arg \frac{z}{\xi} < 0\}}$; it is holomorphic on $\{z \in \mathbb{C}^* \mid \pm \arg \frac{z}{\xi} < 0\}$; it satisfies $A^{\pm}(0) = \mathbf{1}_n$; for $z \in \overline{\xi \cdot \mathbb{R}}$ (9.11) and (9.12) hold. Furthermore, for $r \to 0$ the restrictions $A^{\pm}_{|\overline{\xi \cdot \mathbb{R}}}$ tend uniformly to $\mathbf{1}_n$.
The right hand sides of (9.11) and (9.12) are holomorphic in $\pm\widehat{I}(\xi)$. With the theorem of Morera one sees that $A^{\pm}$ extend to holomorphic matrices on $\widehat{I}_{\pm}(\xi)$ (this argument is taken from [Dou83, page 296]).
We still have to show that $(A^{\pm})_0 \in \Gamma(I_{\pm}(\xi), GL(\mathcal{A}))$ and $(A^{\pm})_\infty \in \Gamma(I_{\pm}(\xi), GL(\gamma^*(\overline{\mathcal{A}})))$. But any $\mathcal{O}_{\mathbb{C},0}$-basis of $\mathcal{H}_0$ gives rise to matrices $\widetilde{A}^{\pm} \in \Gamma(I_{\pm}(\xi), GL(\mathcal{A}))$ which also satisfy (9.11) and (9.12) for $z \in \widehat{I}(\xi)$ close to 0. Therefore the matrices $(\widetilde{A}^{\pm})^{-1} \cdot A^{\pm}$ glue to a matrix which is continuous and invertible close to 0 and

holomorphic outside 0. Thus it is holomorphic at 0. Therefore $(A^{\pm})_0 \in \Gamma(I_{\pm}(\xi), GL(\mathcal{A}))$. The same applies at ∞. The Riemann boundary value problem is solved for $r \ll 1$, and $\pi_r^*(H, \nabla, H'_{\mathbb{R}}, P)$ is a pure TERP-structure. It remains to show that it is polarized. Let now $\underline{\omega}$ be a global basis as above. The hermitian pairing is given by the hermitian and z-independent matrix

$$z^{-w} \cdot P(\underline{\omega}^{tr}, \underline{\omega}) = (A^+(z))^{tr} \cdot P_{mat}(r) \cdot \overline{A^-(-\frac{1}{\overline{z}})} = P_{mat}(r) \cdot \overline{A^-(\infty)}. \tag{9.14}$$

The matrix $P_{mat}(r)$ is hermitian and positive definite.
Now one way to conclude is to go through the construction simultaneously for all Stokes matrices $T(t) := (1-t) \cdot T^{model} + t \cdot T$ with $t \in [0,1]$. For sufficiently small r it works, and the matrix $\overline{A^-(\infty)}(t)$ depends continuously on t, with $\overline{A^-(\infty)}(t=0) = \mathbf{1}_n$. Therefore for all $t \in [0,1]$ the hermitian matrix $P_{mat}(r) \cdot \overline{A^-(\infty)}(t)$ is positive definite.

10 Semi-simple case and ADE-singularities

A TERP-structure is called semi-simple if the eigenvalues of the pole part are all different. Such TERP-structures automatically do not require a ramification (e.g. [Sab02, II.5.7]). Furthermore, a semi-simple TERP-structure is determined by very elementary data, as is shown in the following lemma. Let us call two matrices T and T' in $M(n \times n, \mathbb{C})$ sign equivalent if there is a matrix $B = \text{diag}(\pm 1, ..., \pm 1)$ such that $BTB = T'$.

Lemma 10.1. *1. Fix a weight $w \in \mathbb{Z}$, n different values $u_1, ..., u_n \in \mathbb{C}$ and $\xi \in S^1$ with $\Re(\frac{u_i - u_j}{\xi}) < 0$ for $i < j$. There is a natural 1-1 correspondence between the set of semi-simple TEP-structures (i.e. no real structure) of weight w with pole part having eigenvalues $u_1, ..., u_n$, and the set of sign equivalence classes of upper triangular matrices $T \in M(n \times n, \mathbb{C})$ with diagonal entries equal to 1. The matrices T are called Stokes matrices of the TEP-structure.*

2. The correspondence in 1. restricts to a correspondence between semi-simple mixed TERP-structures and sign equivalence classes of matrices with real entries.

3. Given a semi-simple TERP-structure of weight w with eigenvalues $u_1, ..., u_n \in \mathbb{C}$ and $\xi \in S^1$ as above, the corresponding Stokes matrix T is constructed as follows. We use the notations and results of chapter 8, in particular, lemma 8.3. The pieces $\mathcal{L}_{\pm,j}$ of the canonical (after the choice of $\xi \in S^1$) splittings $\mathcal{L}_{|I_{\pm}(\xi)} = \bigoplus_{j=1}^n \mathcal{L}_{\pm,j}$ have rank 1. There are bases $\underline{e}^{\pm} = (e_1^{\pm}, ..., e_n^{\pm})$ of $\mathcal{L}_{|I_{\pm}(\xi)}$ unique up to the common signs of the pairs e_j^+ and e_j^- with the following properties: they are compatible with the splittings, they satisfy $P((\underline{e}^+)^{tr}, \underline{e}^-) = (-i)^w \cdot \mathbf{1}_n$ and $(\underline{e}^-)_{\xi} = (\underline{e}^+)_{\xi} \cdot T$ with $T_{jj} = 1$ for all j. Then T is a Stokes matrix.

Proof. First we prove the last part: If $\underline{e}^{\pm}$ are any bases of $\mathcal{L}_{|I_{\pm}(\xi)}$ which are compatible with the splittings, then T is upper triangular with $\det T \neq 0$ by lemma 8.3, and $P((\underline{e}^+)^{tr}, \underline{e}^-)$ is diagonal and invertible by lemma 8.4. The constraints $T_{jj} = 1$ and $P(e_j^+, e_j^-) = (-i)^w$ determine e_j^+ and e_j^- up to a common sign. This shows part 3.
Now suppose that the semi-simple TEP-structure is a mixed TERP-structure. Real structure and Stokes structure are compatible, thus there exist $\lambda_j^{\pm} \in S^1$ such that $(\lambda_1^{\pm} e_1^{\pm}, ..., \lambda_n^{\pm} e_n^{\pm})$ are real bases of $\mathcal{L}_{|I_{\pm}(\xi)}$. Because of $T_{jj} = 1$ one can choose $\lambda_j^- = \lambda_j^+$. Then $(\lambda_j^+)^2 \cdot (-i)^w = P(\lambda_j^+ e_j^+, \lambda_j^+ e_j^-) \in i^w \cdot \mathbb{R}$ shows $\lambda_j^+ \in \{\pm 1\}$, so $\underline{e}^{\pm}$ are real bases and T has real entries. This proves one direction in 2.
For the other direction and for the first part, we start from $u_1, ..., u_n, \xi$ and T as in 1. Let us first construct the topological data (H', ∇, P), and $H'_{\mathbb{R}}$ in case T is real. Let $\mathcal{L}_{\pm}$ be local systems of rank n on $I_{\pm}(\xi)$ with bases $\underline{e}^{\pm} = (e_1^{\pm}, ..., e_n^{\pm})$. They are glued to a local system $\mathcal{L}$ on S^1 by $(\underline{e}^-)_{\xi} = (\underline{e}^+)_{\xi} \cdot T$ and $(\underline{e}^-)_{-\xi} = (\underline{e}^+)_{-\xi} \cdot (-1)^w T^{tr}$. If T is real then the bases $\underline{e}^{\pm}$ induce a real structure on $\mathcal{L}$. In any case the formulas $P((\underline{e}^+)^{tr}, \underline{e}^-) := (-i)^w \cdot \mathbf{1}_n$ and $P((\underline{e}^-)^{tr}, \underline{e}^+) := i^w \cdot \mathbf{1}_n$ give a well-defined flat $(-1)^w$-symmetric nondegenerate pairing on opposite fibers. This yields the topological data $(H', \nabla, H'_{\mathbb{R}}$ if T is real$, P)$.
In the special case $T = \mathbf{1}_n$ they decompose as $\bigoplus_{j=1}^n (H'_j, \nabla_j, H'_{j,\mathbb{R}}, P_j)$. Then we write $\underline{f}^{\pm}$ instead of $\underline{e}^{\pm}$, and $\mathcal{L}_j$ denotes the local system on S^1 generated by $f_j^{\pm}$. Put $\mathcal{O}(H_j) = \mathcal{H}_j := \mathcal{O}_{\mathbb{C}} \cdot z^{w/2} \cdot f_j^{\pm}$, then $(H_j, \nabla_j, H'_{j,\mathbb{R}}, P_j)$ is a regular singular TERP-structure. Using (2.3), (5.4) and (5.5) one checks that it induces a PHS of weight w if w is even and of weight $w-1$ if w is odd. Therefore $\bigoplus_{j=1}^n e^{-u_j/z} \otimes (H_j, \nabla_j, H'_{j,\mathbb{R}}, P_j)$ is a mixed TERP-structure of weight w with Stokes matrix $T = \mathbf{1}_n$.

It remains to construct, for arbitrary T, an extension $H \in VB_{\mathbb{C}}$ of H' such that (H, ∇, P) is a TEP-structure formally isomorphic to $\bigoplus_{j=1}^n e^{-u_j/z} \otimes (H_j, \nabla_j, P_j)$. The key step is the construction of two invertible matrices $A^\pm \in \Gamma(I_\pm(\xi), GL(\mathcal{A}))$ satisfying

$$(\underline{e}^+ \cdot z^{w/2} \cdot R_0 \cdot A^+)_{|\widehat{I}(\pm\xi)} = (\underline{e}^- \cdot z^{w/2} \cdot R_0 \cdot A^-)_{|\widehat{I}(\pm\xi)}. \tag{10.1}$$

Here $R_0 := \text{diag}(e^{-u_1/z}, ..., e^{-u_n/z})$. Then $\underline{\omega} := \underline{e}^\pm \cdot z^{w/2} \cdot R_0 \cdot A^\pm$ defines an extension $H \in VB_{\mathbb{C}}$ of H' by $\mathcal{H} = \bigoplus_{j=1}^n \mathcal{O}_{\mathbb{C}} \cdot \omega_j$ with all desired properties, namely: It is a semi-simple TEP-structure. The isomorphisms $\Psi_\pm : \mathcal{L}_{|I_\pm(\xi)} \to \bigoplus_{j=1}^n \mathcal{L}_{j|I_\pm(\xi)}$, $e_j^\pm \mapsto f_j^\pm$, extend to isomorphisms as in (8.3) and (8.4). The pairing P satisfies

$$P(\underline{\omega}^{tr}(z), \underline{\omega}(-z)) \quad = \quad z^w \cdot A^\pm(z)^{tr} \cdot A^\mp(-z) \qquad \text{if } z \in \widehat{I}_\pm(\xi), \tag{10.2}$$

the entries of this matrix are elements in $z^w\Gamma(S^1, \mathcal{A}) = z^w\mathbb{C}\{z\}$, and the matrix $z^{-w} \cdot P(\underline{\omega}^{tr}(z), \underline{\omega}(-z))$ is invertible at 0.
The two conditions on $A^\pm$ in (10.2) are equivalent to the two conditions

$$(A^+ \cdot (A^-)^{tr})_{|\widehat{I}(\xi)} \quad = \quad (R_0^{-1} \cdot T \cdot R_0)_{|\widehat{I}(\xi)}, \tag{10.3}$$

$$(A^+ \cdot (A^-)^{tr})_{|\widehat{I}(-\xi)} \quad = \quad (R_0^{-1} \cdot (-1)^w T^{tr} \cdot R_0)_{|\widehat{I}(-\xi)}. \tag{10.4}$$

As in the proof of theorem 9.3 2. one checks that both matrices on the right hand side have the asymptotic development $\mathbf{1}_n$. [Mal83, proposition A.1] applies and gives the existence of $A^\pm$. By construction, T is a Stokes matrix of the TEP-structure (H, ∇, P). This gives the correspondences in 1. and 2. □

Remarks:

1. In many interesting cases the Stokes matrix has actually entries in $\mathbb{Z}$. This holds in singularity theory for the TERP-structures defined by function germs (e.g. [Pha83][Her03, ch. 8]) and by tame functions [Sab][DS03][Sab05a], in quantum cohomology at least for the mirror partners of tame functions, and also in the massive supersymmetric field theories considered in [CV91][CV93].

2. It is a major point in [CV91][CV93] that for field theories considered therein, the quite elementary data $(u_1, ..., u_n, \xi, T)$ in lemma 10.1 2. determine a mixed semi-simple TERP-structure and thus allow to recover most of the geometry of the field theory.

3. Contrary to PMHS, a semi-simple mixed TERP-structure has at most one compatible lattice in $H'_{\mathbb{R}}$, and that exists precisely iff the Stokes matrix has entries in $\mathbb{Z}$.

4. One might ask which of the data $(u_1, ..., u_n, \xi, T)$ in lemma 10.1 2. give rise to a polarized pure TERP-structure. [Dub93, proposition 2.2] resp. theorem 9.3 2. say that it is sufficient to have that all differences $|u_i - u_j|$ $(i \neq j)$ are sufficiently large. The following conjecture proposes another partial answer.

Conjecture 10.2. *Fix $u_1, ..., u_n \in \mathbb{C}$, $\xi \in S^1$ with $\Re(\frac{u_i - u_j}{\xi}) < 0$ for $i < j$ and $T \in M(n \times n, \mathbb{R})$ upper triangular with $T_{ii} = 1$ and such that $T + T^{tr}$ is positive definite. Then the corresponding mixed TERP-structure of weight w is pure and polarized. Its spectral numbers at infinity are in the interval $(\frac{w-1}{2}, \frac{w+1}{2})$.*

Remarks: Actually, the matrix $(-1)^w T^{-1} T^{tr}$ gives the monodromy with respect to the basis $\underline{e}^-$ in lemma 10.1. This matrix leaves invariant the pairing, which is given by the matrix $T + T^{tr}$. Therefore the monodromy is semi-simple with eigenvalues in $S^1 - \{(-1)^{w+1}\}$.
If the conjecture is true then it applies to all TERP-structures in a family $\bigcup_{r>0} \pi_{r^{-1}}^*(H, \nabla, H'_{\mathbb{R}}, P)$ because they have the same Stokes data and the eigenvalues are given by $r \cdot u_1, ..., r \cdot u_n$. Then the TERP-structure $(H, \nabla, H'_{\mathbb{R}}, P)$ induces a Sabbah orbit, theorem 7.3 applies and yields a sum of pure PHS of weight w and $w - 1$.

Theorem 10.3. *The conjecture is true if T arises as follows. Consider a root system of type ADE, its root lattice $L \cong \mathbb{Z}^n$ with pairing $(-, -)$ and any basis of L consisting of roots $\beta_1, ..., \beta_n$ such that $s_{\beta_1} \circ ... \circ s_{\beta_n}$ is a Coxeter element. Let T be the upper-triangular matrix defined by $T_{ii} := 1$ and $T_{ij} := (\beta_i, \beta_j)$ for $i < j$.*

Proof. The proof consists in putting together different results on the ADE-singularities. We proceed in eight steps.

Step 1. There is a standard universal unfolding $F(x,t) = f_t(x) = f_0(x) + \sum_{i=1}^n t_i m_i(x)$ on $\mathbb{C}^w \times \mathbb{C}^n$ of an ADE-singularity f_0 such that $F(x,t)$ is weighted homogeneous in x and t with weighted degrees $\deg F(x,t) = \deg(f_0) = 1$ and $\deg m_i < 1$ (e.g. [Arn72, 8.4][Loo74, (2.2)]). The parameter space is $M = \mathbb{C}^n$. Any function $f_t : \mathbb{C}^w \to \mathbb{C}$ is tame (see the next chapter for this notion).

Step 2. Fix an arbitrary set of n different numbers $u_1, ..., u_n \in \mathbb{C}$. By [Loo74, (2.4)] the number of parameters $t \in \mathbb{C}^n$ such that the critical values of f_t are $u_1, ..., u_n$ is $(n+1)^{n-1}$ for A_n, $2(n-1)^n$ for D_n, $2^9 \cdot 3^4$ for E_6, $2 \cdot 3^{12}$ for E_7 and $2 \cdot 3^5 \cdot 5^7$ for E_8.

Step 3. By [Pha85][Her03, ch. 8] the oscillating integrals of any function f_t in the universal unfolding induce a TERP-structure of weight w. It does not require a ramification, and the regular singular pieces are essentially the Brieskorn lattices of the local singularities. Because they induce PMHS ([Var80][SS85][Sai89], for the polarization see [Her02]), it is a mixed TERP-structure. The eigenvalues of the pole part are the critical values of f_t.

Step 4. By [DS03] the TERP-structures for f_t, $t \in M$, fit together to a variation of TERP-structures. Because of $\deg_w m_i < 1 = \deg f_0$, the spectral numbers at infinity (and also the filtration $\widetilde{F}^\bullet_{Sab}$) are constant. At $t = 0$ they coincide with the usual spectral numbers. They lie in the interval $(\frac{w-1}{2}, \frac{w+1}{2})$.

Step 5. By [Sab05a, theorem 4.9] any tame function on an affine manifold gives rise to a polarized pure TERP-structure via its oscillating integrals. This applies to the TERP-structure of f_t.

Step 6. Fix n different numbers $u_1, ..., u_n \in \mathbb{C}$ and $\xi \in S^1$ with $\Re(\frac{u_i - u_j}{\xi}) < 0$ for $i < j$. Choose a function f_t with critical values $u_1, ..., u_n$.

Claim: *A Stokes matrix T (of the sign equivalence class in lemma 10.1) of the TERP-structure of f_t takes the following form: $(-1)^{(w-1)(w-2)/2} \cdot (T - \mathbf{1}_n)$ is the strictly upper triangular part of the intersection matrix of a certain distinguished basis $\underline{\delta}$ of the Milnor lattice $H_{w-1}(f_t^{-1}((-i)\xi \cdot r), \mathbb{Z})$ for some $r \gg 0$. The distinguished basis $\underline{\delta}$ is constructed below.*

For the construction of $\underline{\delta}$, we choose $r \gg \max |u_i|$. We connect u_i with $u_i + (-i)\xi \cdot r$ by a straight line and $u_i + (-i)\xi \cdot r$ with $(-i)\xi \cdot r$ by a straight line. Along this path from u_i to $(-i)\xi \cdot r$ the vanishing cycle of the simple critical point of f_t with value u_i is shifted to the Milnor fiber $H_{w-1}(f_t^{-1}((-i)\xi \cdot r), \mathbb{Z})$. This gives a distinguished basis $\underline{\delta} = (\delta_1, ..., \delta_n)$ of $H_{w-1}(f_t^{-1}((-i)\xi \cdot r), \mathbb{Z})$ [AGZV88, pages 14 and 31]. It is unique up to the signs $\pm\delta_i$.
For the proof of the claim one needs several facts and formulas:

(i) The bundle H' of the TERP-structure is the bundle dual to the bundle of Lefschetz thimbles, and $P = (-1)^{(w-1)w/2} \cdot \frac{1}{(2\pi i)^w} \cdot P^*_{Lef}$ where P_{Lef} is the intersection form for Lefschetz thimbles, see [Her03, ch. 8].

(ii) The Stokes structure is determined by the Lefschetz thimbles along $\bigcup_{i=1}^n (u_i + (-i)\xi \cdot \mathbb{R}_{>0})$ and the Lefschetz thimbles along $\bigcup_{i=1}^n (u_i + i\xi \cdot \mathbb{R}_{>0})$.

(iii) One needs the precise relations between Lefschetz thimbles, vanishing cycles, their intersection forms and the Seifert form, see [AGZV88, ch. 2].

We omit the details of the proof of the claim. The claim is consistent with the conventions in [AGZV88]. The Picard-Lefschetz transformations $s_{\delta_i} = s_{-\delta_i}$ satisfy $s_{\delta_1} \circ ... \circ s_{\delta_n} =$ monodromy.

Step 7. Fix $u_1, ..., u_n \in \mathbb{C}$, $\xi \in S^1$ and $r \gg 0$ as in step 6 and denote

$$\text{Par} := \{t \in \mathbb{C}^n \mid f_t \text{ has critical values } u_1, ..., u_n\}.$$

The Milnor lattices $H_{w-1}(f_t^{-1}((-i)\xi \cdot r), \mathbb{Z})$ for $f_t \in \text{Par}$ are canonically isomorphic.
Now suppose that w is odd. It is well known (e.g. [Arn72]) that in that case an isomorphism $H_{w-1}(f_t^{-1}((-i)\xi \cdot r), \mathbb{Z}) \stackrel{\cong}{\to} L$ exists which maps the intersection form to $(-1)^{(w-1)/2}(-,-)$ and the monodromy to a Coxeter element c in the Weyl group of L.

By [Del, §2] this isomorphism identifies the set of all distinguished bases of the Milnor lattice (definition in [AGZV88, pages 14 and 31]) with the set $\mathcal{B} := \{(\beta_1, ..., \beta_n) \mid \beta_i \in L, (\beta_i, \beta_i) = 2, s_{\beta_1} \circ ... \circ s_{\beta_n} = c\}$, and the natural braid group action on this set is transitive.
Define an equivalence relation on $\mathcal{B}$ by $\underline{\beta} \sim \underline{\beta}' \iff (\beta_1, ..., \beta_n) = (\pm\beta_1', ..., \pm\beta_n')$. Denote by $\underline{\delta}(t) \in \mathcal{B}$ the distinguished basis of f_t, $t \in \mathrm{Par}$, which was constructed in step 6. In [Del, §3] the formula $|\mathrm{Par}| = |\mathcal{B}/\sim|$ is proved. This and the discussion in [Loo74, §3] show that the map

$$\mathrm{Par} \to \mathcal{B}/\sim, \quad t \mapsto [\underline{\delta}(t)] \tag{10.5}$$

is a bijection.

Step 8. The steps 7 and 6 show that any matrix T in theorem 10.3 is realized as Stokes matrix of the TERP-structure of a function f_t with $t \in \mathrm{Par}$. The steps 1 to 7 show theorem 10.3 for odd w. The case of the even number $w+1$ can be treated by twisting the TERP-structure with $z^{1/2}$ or by using the formulas in [AGZV88, 2.8] for the intersection forms of $f_t(x)$ and $f_t(x) + x_{w+1}^2$ with respect to distinguished bases. □

Remarks:

1. The number of parameters in Par resp. of equivalence classes of distinguished bases with fixed sign equivalence class of Stokes matrices is larger than one. It is $n+1$ for A_n, $2(n-1)$ for D_n, 12 for E_6, 9 for E_7 and 15 for E_8. This follows from [Voi85][Klu89]. There the numbers of distinguished bases with fixed Coxeter-Dynkin diagrams are listed. It turns out to be twice the number above: The Stokes matrices correspond to the Coxeter-Dynkin diagrams of the distinguished bases. In one equivalence class in $\mathcal{B}/\sim$ exactly two distinguished bases have the same Coxeter-Dynkin diagram, $(\beta_1, ..., \beta_n)$ and $(-\beta_1, ..., -\beta_n)$. This follows from the connectedness of the Coxeter-Dynkin diagrams.

   The number above is also the order of the automorphism group R_{f_0} studied in [Her02, 13.2].

2. The results on Landau-Ginzburg models in [CV91][CV93] are closely related to the crucial step 5, [Sab05a, theorem 4.9]. The ADE case follows also from [CV91][CV93]. But the proofs are completely different.

Proposition 10.4. *Conjecture 10.2 is true in the rank two case.*

Proof. We only sketch the proof. In [Dub93] and [CV91][CV93] it is proved that there is a correspondence between semi-simple TERP-structures $(H, \nabla, H'_{\mathbb{R}}, P)$ such that $\pi^*_{r^{-1}}(H, \nabla, H'_{\mathbb{R}}, P)$ is pure polarized for all $r > 0$ and real smooth solutions on $(0, \infty)$ of the sinh-Gordon equation $(\partial_r^2 + \frac{1}{r}\partial_r)u(r) = \sinh u(r)$. It also follows implicitly from [IN86]. The hermitian metric with respect to a certain basis is given by the matrix

$$\begin{pmatrix} \cosh(\frac{u(r)}{2}) & -i\sinh(\frac{u(r)}{2}) \\ i\sinh(\frac{u(r)}{2}) & \cosh(\frac{u(r)}{2}) \end{pmatrix}.$$

In [MTW77] a family depending on one real parameter of such solutions has been studied. In [IN86, ch. 11] it is shown that the Stokes matrices of these solutions are $T = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ with $t \in (-2, 2)$; it is the case $A = B$ and $Q = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$ in [IN86, ch. 11]. This shows the first claim of the conjecture.
The behavior of the solutions for $r \to 0$ [MTW77][IN86, (11.2)] shows that the spectral numbers at infinity are in the interval $(-\frac{1}{2}, \frac{1}{2})$. This shows the conjecture for $w = 0$. For other w one twists the TERP-structure with $z^{w/2}$. □

Remarks: Also singular solutions of the sinh-Gordon equation correspond to families $\pi^*_{r^{-1}}(H, \nabla, H'_{\mathbb{R}}, P)$ of semi-simple TERP-structures of rank 2. At a singularity $r_k > 0$ a solution has the asymptotic form $u(r) = -2\log(r - r_k) - O(r - r_k)$ [IN86, (11.6)]. On one side of r_k it is real, on the other side it takes values in $2\pi i + \mathbb{R}$. Then $\pi^*_{r_k^{-1}}(H, \nabla, H'_{\mathbb{R}}, P)$ is not pure, and the hermitian form defined by $\pi^*_{r^{-1}}(H, \nabla, H'_{\mathbb{R}}, P)$ is positive definite for r on one side and negative definite for r on the other side.
In [MTW77] also the solutions corresponding to Stokes matrices $T = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ with $t \in \mathbb{R} - [-2, 2]$ are studied. By theorem 9.3 they are smooth for $r \to \infty$. But they have infinitely many singularities for $r \to 0$. Their

distribution is studied in [MTW77, page 1090]. Conjecture 9.2 predicts that all solutions with Stokes matrix $T = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ with $t \in \mathbb{C}\backslash\mathbb{R}$ have singularities. In [IN86, ch. 11] only solutions which are smooth for small r with asymptotics [IN86, (11.2)] are studied. For those the rank two case of the conjecture 9.2 is proved: Only the solutions in the one parameter family of [MTW77] do not have singularities. All other solutions have infinitely many singularities $r_k = \pi(k - \frac{1}{2}) + O(\log k)$, $k \to \infty$ [IN86, (11.10)].
So in the semi-simple cases of rank two which are studied, there are infinitely many singularities, i.e. not pure TERP-structures, if there are any singularities at all. This is in sharp contrast to the regular singular rank two cases discussed in [Her03, 8.3], where the TERP structures in the families $\bigcup_{r>0} \pi_{r^{-1}}(H, \nabla, H'_{\mathbb{R}}, P)$ are pure everywhere except for one parameter.

11 Remarks on applications

As we already pointed out at several places, one of the major sources of examples of TERP-structures is singularity theory. Oscillating integrals of a holomorphic function with isolated singularities give rise to a TERP-structure. Oscillating integrals have been studied since long time ([Pha85][AGZV88] and references therein). It is well known that they arise by a Fourier transformation from the Gauss-Manin system and the Brieskorn lattice.
The map τ extending the bundle H to a bundle $\widehat{H}$ on $\mathbb{P}^1$ was first considered for the TERP-structures from oscillating integrals in the work of Cecotti and Vafa [CV91][CV93] on Landau-Ginzburg models. These papers were a source of inspiration for [Her03] and also for the present article. A similar construction, but using a hermitian metric from the very beginning instead of a real structure and a pairing P, is present in the work of Simpson on harmonic bundles [Sim90][Sim92][Sim97]. Recently, this theory was generalized by Sabbah [Sab04] to what he calls polarizable twistor $\mathcal{D}$-modules.
Let us give a very brief summary on the situation in singularity theory leading to TERP-structures. We consider simultaneously two cases, which are called local and global:
Local case. $f : (\mathbb{C}^w, 0) \to (\mathbb{C}, 0)$ is a holomorphic function germ with an isolated singularity at 0 with Milnor number n.
Global case. $f : Y \to \mathbb{C}$ is a function on an affine manifold Y of dimension w such that f is M-tame (definition in [NS99] and [DS03, 2.a]) and cohomologically tame (definition in [Sab]). We simply call such f tame. Then f has only isolated singularities. The sum n of the Milnor numbers of the singularities is the (global) Milnor number of f.
In both cases there exists a semi-universal unfolding F (where in the global case "semi-universal" refers to the Kodaira-Spencer map being an isomorphism, see [DS03]) with smooth base space M, isomorphic to the germ $(\mathbb{C}^n, 0)$. The deformed functions F_t for $t \in M$ are defined on a some small or large Stein manifold (in the local case a small ball in $\mathbb{C}^w$). In the global case here M-tameness is used [DS03].

Theorem 11.1. *In both cases one obtains a variation of mixed TERP-structures $\bigcup_{t\in M}$ TERP(F_t) of rank n on M. The notation TERP$(F_t) = (H(t), \nabla, H'_{\mathbb{R}}, P)$ makes sense because the topological data $(H' = H_{|\mathbb{C}^*}, \nabla, H'_{\mathbb{R}}, P)$ are canonically isomorphic for all TERP-structures TERP(F_t). H' is the bundle dual to the bundle of Lefschetz thimbles, ∇ is the natural flat connection from shifting Lefschetz thimbles. The pairing P is defined up to a constant by the intersection form of Lefschetz thimbles. The sections in $\mathcal{O}(H)$ come from the Fourier transform of the Gauss-Manin system.*

The precise construction in the local case is described in [Her03, 8.1]. In the global case it is given for $F_0 = f$ in [Sab] and for any F_t in [DS03]. It builds on the work of many people on the Gauss-Manin system and the Brieskorn lattice.
The TERP-structure *TERP*(F_t) is a mixed TERP-structure: By construction, it does not require a ramification and the regular singular pieces are essentially the local Brieskorn lattices of the singularities of F_t. Compatibility of real structure and Stokes structure is trivial, because the splittings of the local system comes from topology, from the Lefschetz thimbles and vanishing cycles associated to the singularities. The fact that the regular singular pieces give rise to PMHS is due to [Var80][SS85] (for the polarization see [Her02, ch. 10]).
The Landau-Ginzburg models of Cecotti and Vafa [CV91][CV93] involve a lot of additional structure from physics, but the central objects of study are the TERP-structures of tame functions $f : Y \to \mathbb{C}$ (or at least a substantial subfamily of them). From physical considerations Cecotti and Vafa derive the following. An

independent completely different purely mathematical proof was given recently by Sabbah [Sab05a, theorem 4.9][Sab04].

Theorem 11.2. *The TERP-structure of a tame function $f : Y \to \mathbb{C}$ on an affine manifold Y is pure and polarized.*

In [CV91][CV93] the resulting positive definite hermitian metric h is their ground state metric. This fundamental theorem will certainly play an important role in the future study of tame functions. It can be considered as the analogue for tame functions of the fact that the (primitive part of the) cohomology of a compact Kähler manifold carries (polarized) Hodge structures.
If we consider a semi-universal unfolding F of a tame function $f : Y \to \mathbb{C}$, then most of the functions F_t, $t \in M$, have to be restricted to a Stein subset of Y, as they would have additional singularities "from infinity" on Y. Therefore the theorem applies only to a certain subfamily of all functions in the semi-universal unfolding, the subfamily along which the global Milnor number n and the tameness condition are preserved.
Let us discuss some applications of the main results of this paper (theorems 7.3 and 9.3, the latter containing theorem 6.6) to TERP-structures coming from local and global singularities. First we state a simple lemma.

Lemma 11.3. *Consider a function F_t in a semi-universal unfolding of a function f, in the local or the global case. Then for any $r \in \mathbb{C}^*$*

$$TERP(r \cdot F_t) = \pi^*_{r^{-1}}(TERP(F_t)). \tag{11.1}$$

Furthermore, the one parameter unfolding $r \cdot F_t$, $r \in \mathbb{C}^*$, of F_t is (for r close to 1) isomorphic to the Euler field orbit of F_t in the universal unfolding, and the pull back of the Euler field is the vector field $r\partial_r$ on $\mathbb{C}^*$.

Proof. The first part follows from the formulas for the Fourier transformation of the Gauss-Manin system. The central point is $\pi^*_{r^{-1}}(e^{-\theta/z}) = e^{-r\cdot\theta/z}$; here θ is the variable for the values of F_t. The second part is proved in [Her03, lemma 8.6]. The central formula is $r\partial_r(r \cdot F_t) = r \cdot F_t$. □

The next result gives two major applications of our correspondence. It shows nicely that both implications of the correspondence are of interest.

Corollary 11.4. *1. $TERP(F_t)$ induces a nilpotent orbit. For $|r| \gg 0$ the TERP-structure $TERP(r \cdot F_t)$ is pure and polarized.*

2. In the case $F_0 = f : Y \to \mathbb{C}$ tame, $TERP(f)$ induces a Sabbah orbit. Sabbah's Hodge filtration makes the tuple $(H^\infty, H^\infty_{\mathbb{R}}, N, S, \widetilde{F}^\bullet_{Sab})$ (see theorem 7.3) into a PMHS of weight $w-1$ resp. w on $H^\infty_{\neq 1}$ resp. H^∞_1.

Proof. 1. $TERP(F_t)$ is a mixed TERP-structure by theorem 11.1. The second part of theorem 9.3 shows that it a nilpotent orbit. Now formula (11.1) gives the second claim.

2. $r \cdot f$ is tame for any $r \in \mathbb{C}^*$. By theorem 11.2 $TERP(r \cdot f)$ is pure and polarized for any $r \in \mathbb{C}^*$. In particular, $TERP(f)$ induces a nilpotent as well as a Sabbah orbit. Now theorem 7.3 applies.

□

The first point of the corollary proves the main part of conjecture 8.3 in [Her03]. The second part strengthens a former result of Sabbah. Namely, it was shown in [Sab] (building on [Sab97]) that $(H^\infty, H^\infty_{\mathbb{R}}, W_\bullet, F^\bullet_{Sab})$ is a mixed Hodge structure, where $W_\bullet$ is the weight filtration from the nilpotent part N of the monodromy and $F^\bullet_{Sab}$ is the filtration on H^∞ defined in (7.1). Polarizations are not considered in these papers. To obtain a PMHS, we need to work with the twisted filtration $\widetilde{F}^\bullet_{Sab} := G^{-1}(F^\bullet_{Sab})$, but as already said, it coincides with $F^\bullet_{Sab}$ on the quotient of the weight filtration.

Remarks: It is interesting to ponder further on the logical interrelations.

- We know already that the TERP-structure $TERP(f)$ for $f : Y \to \mathbb{C}$ is mixed. But Theorem 11.2 and the unproved direction of conjecture 9.2 would give it again, because theorem 11.2 applied to all $r \cdot f$ shows that $TERP(f)$ induces a nilpotent orbit. This would give a new proof that Stokes structure and real structure are compatible and that the Brieskorn lattices of the singularities of f induce PMHS.

- The other way round, the first part of corollary 11.4 applied to tame $f : Y \to \mathbb{C}$ shows that theorem 11.2 is true for $r \cdot f$ with $r \gg 0$.

We conclude with some remarks on quantum cohomology. The following ideas and speculations suggest that using mirror symmetry, theorem 11.2 it is not so far away from the classical fact that the cohomology of Kähler manifolds carries Hodge structures.
Mirror symmetry predicts that certain tame functions are related to certain Fano manifolds, more precisely, that the TEP-structures of the functions are isomorphic to the TEP-structures in the structure connections of the quantum cohomology of the Fano manifolds.
By these isomorphisms, the natural real structures on the singularity side induce real structures on the quantum cohomology side. However, there seems to be for the moment no intrinsic mathematical description of this real structure on the quantum cohomology side (possibly the structures in [CV91][CV93]) have to be studied carefully).
If there were such a description, one could hope that the TERP-structures from the quantum cohomology of any manifold would be pure and polarized. This would probably only hold for the TERP-structures for parameters in or close to the small quantum cohomology, because in some known examples of mirror symmetry small quantum cohomology corresponds to the tame unfoldings F_t of $F_0 = f$.
The classical cohomology of a Kähler manifold endowed with cup product and the sum of Hodge structures is obtained as semiclassical limit from the quantum cohomology ring. In some known examples the semiclassical limit can be considered as the center of a normal crossing divisor along which the family of TEP-structures from quantum cohomology has logarithmic poles. If the quantum multiplication could be used to define a TERP-structures, one could hope to obtain a variation of pure polarized TERP-structures on the complement of the divisor. This may allow to get the classical Hodge structures as a limit PMTS (or a quotient of it by some filtration, also an additional twist is possible) using results from [Moc07] or [Sab05b].

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